

F1812

12

29. 1983.

BULETINUL INSTITUTULUI POLITEHNIC DIN IAȘI

Tomul XXIX (XXXIII)

Fasc. 1 — 4



Secția I

MECANICĂ

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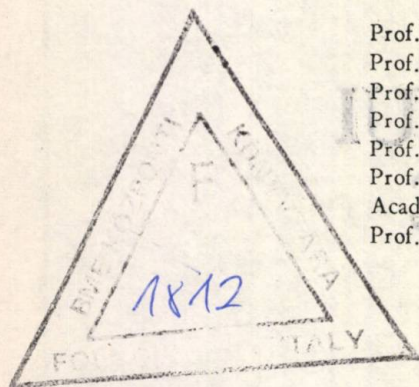
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THE RHEOLOGY OF THERMAL STRESS RELAXATION PROCESS IN METAL

BY

I. HOPULELE and N. SCÂNTEIANU

1. Introduction. Many observations on stressed pieces as a result of a processing procedure led to the conclusion that the diminution of the stress during the thermal stress relaxation treatment could be attributed not only to the relaxation process, but also to the decrease both of the yield point and of the modulus of elasticity, at the same time with an increase in temperature. That is why the approach to this problem must be done by means of a specific theory.

Knowing the factors which determine the induced stress of a metal at a certain temperature, a functional dependence between these factors can be found out, in such a way that the real relaxation phenomenon should be reflected. The parameters which establish the induced stress of a metallic body subjected to a heating cycle are the variation of the yield point with temperature and the variation of the tranquil flow characteristic with temperature.

The functional dependence between these parameters can be established only on the basis of some computation hypotheses: the adjustment of a rheological model or the finding of a functional dependence between the factors modifying the $\sigma(\epsilon)$ diagram during both the heating and the plastic yield creep processes.

2. Rheological Model of the Metal Subjected to Thermal Stress Relaxation Process. The use of a simple rheological model for metal to study the thermal stress relaxation process has as an advantage the possibility of building complex rheological models, so that the metal behaviour under the influence of stress and temperature should be conveyed.

By using only those three basic composition elements of the rheological models (Fig. 1), a complex model, which satisfies as many properties of the real body as possible, can be found. Thus for a steel body both the $\sigma(\epsilon)$ diagram, — even in a simplified form, — and the diagram of the steel stress behaviour in time (creep, relaxation) must be satisfied.

In the case of steel, the rheological model must contain groups of elements so combined as to convey the properties of elasticity, plasticity with cold-hardening, and creep.

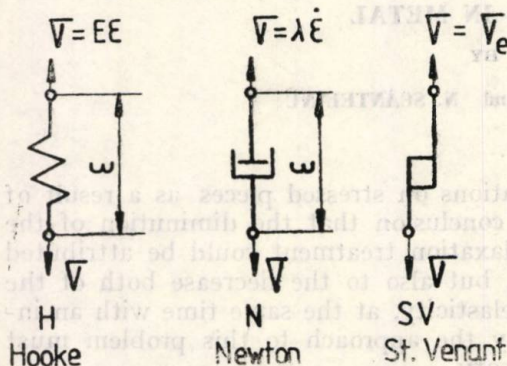


Fig. 1

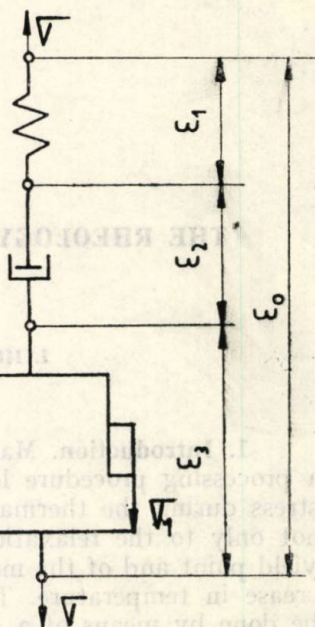


Fig. 2

A rheological model suggested for the study of thermal stress relaxation process, which contains all the above mentioned properties, is presented in Fig. 2; the elastic component produces an ε_1 strain, the creep component an ε_2 strain and the plastic deformation component an ε_3 strain.

The overall strain ε_0 remains constant during the stress relaxation process as the dimensions of the piece are not changed.

On analyzing the model from Fig. 2, we have

$$(1) \quad \varepsilon_0 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3,$$

$$(2) \quad \varepsilon_1 = \frac{\sigma}{E},$$

$$(3) \quad \varepsilon_2 = \frac{1}{\lambda_2} \sigma,$$

$$(4) \quad \dot{\varepsilon}_3 = \frac{1}{\lambda_3} (\sigma - \sigma_1).$$

By differentiating the relation (1) with respect to time, and making the necessary substitutions, we obtain

$$(5) \quad 0 = \dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3,$$

$$(6) \quad 0 = \frac{\sigma E - \dot{E} \sigma}{E^2} + \frac{1}{\lambda^2} \sigma + \frac{1}{\lambda_3} (\sigma - \sigma_1),$$

$$(7) \quad \dot{\sigma} + \left(\frac{E}{\lambda_2} + \frac{E}{\lambda_3} - \frac{\dot{E}}{E} \right) \sigma - \frac{E}{\lambda_3} \sigma_1 = 0,$$

$$(8) \quad \sigma = e \int_0^t \left(\frac{E}{\lambda_2} + \frac{E}{\lambda_3} - \frac{\dot{E}}{E} \right) dt \left[\int_0^t \frac{E}{\lambda_3} \sigma_1 e^{-\int_0^t \left(\frac{E}{\lambda_2} + \frac{E}{\lambda_3} - \frac{\dot{E}}{E} \right) dt} dt + C \right].$$

In the case of thermal stress relaxation process, the C constant is determined stipulating that for $t=0$ we should get $\sigma=\sigma_0$.

In the case of an imposed thermal cycle $T(t)$ for the thermal stress relaxation it is necessary to know the variation of the properties of the rheological model with temperature: $E(T)$, $\lambda_2(T)$, $\lambda_3(T)$ and $\sigma_1(T)$ respectively.

For the examined metal, E is the modulus of elasticity while σ_1 is the technical yield point σ_e ; the viscosity coefficients (λ_2 and λ_3) must be experimentally determined or drawn out from the diagram reflecting the metal properties. Thus λ_2 can be determined by creep tests or stress relaxation while λ_3 by tensile tests.

The functional dependence between all the factors which participate in the change of the stress σ in the piece subjected to a cycle of thermal stress relaxation treatment $T(t)$ is reflected in the relation (8).

3. Conclusions. The rheological model and the relation corresponding to the model are of a real use either for optimizing the technological process of thermal stress relaxation or for establishing a thermal cycle of stress relaxation when a certain final residual stress is required.

Received November 5, 1983

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REOLOGIA PROCESULUI DE DETENSIONARE TERMICĂ A METALULUI

(Rezumat)

Lucrarea are drept scop stabilirea unui model reologic în vederea detensionării termice a produselor metalice în care în urma diferitelor proceselor tehnologice de prelucrare s-au acumulat tensiuni interne.

Folosirea unui model reologic simplu pentru produsul metalic, în vederea studiului procesului de detensionare termică, prezintă avantajul că după el se pot construi modele reologice complexe care să poată reda comportarea produsului metalic sub influența efortului și a temperaturii.

Atît modelul reologic propus cit și relația corespunzătoare modelului reologic sînt de un real folos pentru optimizarea procesului de detensionare termică sau pentru stabilirea unui ciclu termic de detensionare cînd se cere o anumită tensiune reziduală în produsul metalic.

A SECOND ORDER APPROXIMATION FOR EXPECTED VALUE AND VARIANCE TO STUDY KINEMATICAL CHAINES TOLERANCES

BY

RADU FLORESCU

1. Introduction. In the last years the propagation of kinematical chaines deviations is studied more and more using probabilities theory and mathematical statistics. It is a problem of propagation of a random character from the input variables to the output ones, i.e. it is necessary to specify a function of random variables.

In [1], [2] and [3] are indicated formulae for approximate calculation of the expected value and variance for functions of random variables. They are obtained by expanding these functions in a Taylor series in the vicinity of the point in which the independent variables attain their expected values and retaining only the first order terms.

The present Note proposes a better approximation for the above-mentioned moments, by retaining both the first and the second order term, in the Taylor series. It results two expressions for the expected values applicable to functions of independent and dependent random variables respectively and also a formula for the variance of a function of dependent random variables.

We offer an example for how to use our formulae to a slider-crank mechanism.

2. Notation and Definitions. X_i — random variable (r.v.), whose values are x_i ; $f_{X_i}(x)$ — probability density function of X_i ; $Y = g(X_1, \dots, X_i, \dots, X_n)$ — random variable, deterministic function of random variables X_i ($i = \overline{1, n}$), whose values are $y = g(x_1, \dots, x_i, \dots, x_n)$; $E(X)$ — expected value of r.v. X ;

$E_k[X]$ — k -th moment of r.v. X

$$E_k[X] = \int_{-\infty}^{\infty} x^k f_{X_i}(x) dx, \quad E_k[X] = E[X^k], \quad E[X] = E_1[X];$$

$D^2[X]$ — variance of r.v. X

$$D^2[X] = E_2[X] - (E[X])^2;$$

$m_k[X]$ — k -th central moment of r.v. X

$$m_k[X] = E[(X - E[X])^k], \quad D^2[X] = m_2[X];$$

$\text{cov}(X_i, X_j)$ — covariance of r.v. X_i and X_j

$$\text{cov}(X_i, X_j) = E[(X_i - E[X_i])(X_j - E[X_j])];$$

$N: (m_{X_i}, \sigma_{X_i}^2)$ — normal distribution with the expected value m_{X_i} and the variance $\sigma_{X_i}^2$.

3. First Order Approximation for the Expected Value and Variance. If the function

$$Y = g(X_1, \dots, X_i, \dots, X_n)$$

is linearized in the vicinity of the point in which the arguments attain their expected values, [2] and [3] are proposing the following formulae for the expected value and variance:

$$\begin{aligned} E[Y] &\approx g(E[X_1], \dots, E[X_i], \dots, E[X_n]), \\ D^2[Y] &\approx \sum_{i=1}^n \left(\frac{\partial g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i} \Big|_{x_i=E[X_i]} \right)^2 D^2[X_i] + \\ (1) \quad &+ \sum_{j=1}^n \sum_{i \neq j} \left(\frac{\partial g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i} \Big|_{x_i=E[X_i]} \right) \left(\frac{\partial g(x_1, \dots, x_j, \dots, x_n)}{\partial x_j} \Big|_{x_j=E[X_j]} \right) \text{cov}(X_i, X_j). \end{aligned}$$

They are used if the r.v. X_i ($i=1, n$) are dependent ones; if X_i are independent, then [4]

$$(2) \quad \text{cov}(X_i, X_j) = 0.$$

4. The Proposed Second Order Approximation for the Expected Value and Variance. A n -dimensional real valued function $y = g(x_1, \dots, x_i, \dots, x_n)$ of class C^{n+1} may be written as [5]

$$\begin{aligned} y &= g(E[X_1], \dots, E[X_i], \dots, E[X_n]) + \\ (3) \quad &+ \sum_{i=1}^n \left(\frac{\partial g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i} \Big|_{x_i=E[X_i]} \right) (x_i - E[X_i]) + \\ &+ \frac{1}{2} \sum_{i=1}^n \left(\frac{\partial^2 g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i^2} \Big|_{x_i=E[X_i]} \right) (x_i - E[X_i])^2 + \\ &+ \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial^2 g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i \partial x_j} \Big|_{x_i=E[X_i]} \right) (x_i - E[X_i])(x_j - E[X_j]) + R_3, \end{aligned}$$

where R_3 is the remaining term.

Assuming that R_3 can be neglected, according to the properties of the expected value and variance [4], we have

$$\begin{aligned}
 E[Y] &\approx g(E[X_1], \dots, E[X_i], \dots, E[X_n]) + \\
 (4) \quad &+ \frac{1}{2} \sum_{i=1}^n \left(\frac{\partial^2 g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i^2} \Big|_{x_i=E[X_i]} \right) D^2[X_i] + \\
 &+ \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n \left(\frac{\partial^2 g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i \partial x_j} \Big|_{x_i=E[X_i]} \right) \text{cov}(X_i, X_j).
 \end{aligned}$$

If the r.v. X_i ($i=\overline{1, n}$) are independent, by virtue of (2)

$$\begin{aligned}
 E[Y] &\approx g(E[X_1], \dots, E[X_i], \dots, E[X_n]) + \\
 (4') \quad &+ \frac{1}{2} \sum_{i=1}^n \left(\frac{\partial^2 g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i^2} \Big|_{x_i=E[X_i]} \right) D^2[X_i].
 \end{aligned}$$

In order to determine the variance of Y with R_3 negligible, we consider only the case of independent r.v. X_i ($i=\overline{1, n}$):

$$\begin{aligned}
 D^2[Y] &\approx \sum_{i=1}^n \left(\frac{\partial g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i} \Big|_{x_i=E[X_i]} \right) D^2[X_i] + \\
 (5) \quad &+ \frac{1}{4} \sum_{i=1}^n \left(\frac{\partial^2 g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i^2} \Big|_{x_i=E[X_i]} \right)^2 [m_4[X_i] - (D^2[X_i])^2] + \\
 &+ \sum_{i=1}^n \sum_{\substack{j=1 \\ i \neq j}}^n \left(\frac{\partial^2 g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i \partial x_j} \Big|_{x_i=E[X_i]} \right) D^2[X_i] D^2[X_j] + \\
 &+ \sum_{i=1}^n \left(\frac{\partial g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i} \Big|_{x_i=E[X_i]} \right) \left(\frac{\partial^2 g(x_1, \dots, x_i, \dots, x_n)}{\partial x_i^2} \Big|_{x_i=E[X_i]} \right) m_3[X_i].
 \end{aligned}$$

5. Example. We present an example to prove the advantage of (4), (4') and (5) with respect to (1).

Let be the slider-crank mechanism shown in Fig. 1, for which we consider the deviations Δx_i , Δy_i in the joints A , B and C .

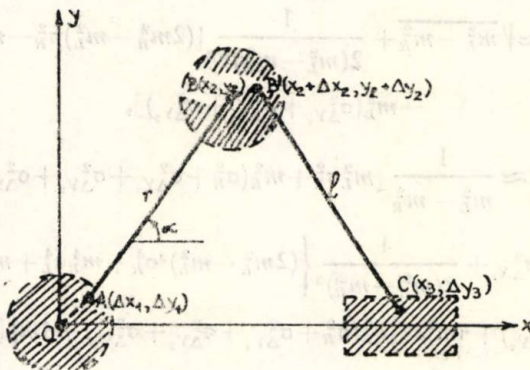


Fig. 1. Slider-crank mechanism. The lined areas suggest the tolerance fields.

Is simply to find

$$(6) \quad x_3 = \Delta x_1 + \Delta x_2 + r \cos \alpha + \sqrt{l^2 - (\Delta y_1 + \Delta y_2 - \Delta y_3 + r \sin \alpha)^2}.$$

In accordance with [1,] [2], [4], all dimensions of the mechanism are considered as normally distributed random variables, i.e. :

$$R : N(m_R, \sigma_R^2); \quad L : N(m_L, \sigma_L^2); \quad \Delta X_i : N(m_{\Delta X_i}, \sigma_{\Delta X_i}^2), \quad i=1, 2;$$

$$\Delta Y_i : N(m_{\Delta Y_i}, \sigma_{\Delta Y_i}^2), \quad i=1, 3.$$

The classical method (formulae (1)) gives :

i) when $\alpha=0$

$$(7) \quad \begin{aligned} E[X_3] &\approx m_R + m_L, \\ D^2[X_3] &\approx \sigma_R^2 + \sigma_L^2 + \sigma_{\Delta X_1}^2 + \sigma_{\Delta X_2}^2; \end{aligned}$$

ii) when $\alpha = \frac{\pi}{2}$

$$(7') \quad \begin{aligned} E[X_3] &\approx \sqrt{m_L^2 - m_R^2}, \\ D^2[X_3] &\approx \frac{1}{m_L^2 - m_R^2} [m_L^2 \sigma_L^2 + m_R^2 (\sigma_R^2 + \sigma_{\Delta Y_1}^2 + \sigma_{\Delta Y_2}^2 + \sigma_{\Delta Y_3}^2)] + \sigma_{\Delta X_1}^2 + \sigma_{\Delta X_2}^2. \end{aligned}$$

The proposed method (formulae (4') and (5)) gives :

i) when $\alpha=0$

$$(8) \quad \begin{aligned} E[X_3] &\approx m_R + m_L - \frac{1}{2m_L} (\sigma_{\Delta Y_1}^2 + \sigma_{\Delta Y_2}^2 + \sigma_{\Delta Y_3}^2), \\ D^2[X_3] &\approx \sigma_R^2 + \sigma_L^2 + \sigma_{\Delta X_1}^2 + \sigma_{\Delta X_2}^2 + \frac{1}{2m_L^2} [\sigma_{\Delta Y_1}^4 + \sigma_{\Delta Y_2}^4 + \sigma_{\Delta Y_3}^4 + \\ &\quad + 4(\sigma_{\Delta Y_1}^2 \sigma_{\Delta Y_2}^2 + \sigma_{\Delta Y_1}^2 \sigma_{\Delta Y_3}^2 + \sigma_{\Delta Y_2}^2 \sigma_{\Delta Y_3}^2)]; \end{aligned}$$

ii) when $\alpha = \frac{\pi}{2}$

$$(8') \quad \begin{aligned} E[X_3] &\approx \sqrt{m_L^2 - m_R^2} + \frac{1}{2(m_L^2 - m_R^2)^{3/2}} [(2m_R^2 - m_L^2) \sigma_R^2 - m_R^2 \sigma_L^2 - \\ &\quad - m_L^2 (\sigma_{\Delta Y_1}^2 + \sigma_{\Delta Y_2}^2 + \sigma_{\Delta Y_3}^2)], \\ D^2[X_3] &\approx \frac{1}{m_L^2 - m_R^2} [m_L^2 \sigma_L^2 + m_R^2 (\sigma_R^2 + \sigma_{\Delta Y_1}^2 + \sigma_{\Delta Y_2}^2 + \sigma_{\Delta Y_3}^2)] + \\ &\quad + \sigma_{\Delta X_1}^2 + \sigma_{\Delta X_2}^2 + \frac{1}{2(m_L^2 - m_R^2)^3} \left\{ (2m_R^2 - m_L^2)^3 \sigma_R^4 + m_R^4 \sigma_L^4 + m_L^4 (\sigma_{\Delta Y_1}^4 + \right. \\ &\quad + \sigma_{\Delta Y_2}^4 + \sigma_{\Delta Y_3}^4) + 4m_R^2 m_L^2 \sigma_L^2 (\sigma_R^2 + \sigma_{\Delta Y_1}^2 + \sigma_{\Delta Y_2}^2 + \sigma_{\Delta Y_3}^2) + 4m_L^4 [\sigma_R^2 (\sigma_{\Delta Y_1}^2 + \\ &\quad + \sigma_{\Delta Y_2}^2 + \sigma_{\Delta Y_3}^2) + \sigma_{\Delta Y_1}^2 \sigma_{\Delta Y_2}^2 + \sigma_{\Delta Y_1}^2 \sigma_{\Delta Y_3}^2 + \sigma_{\Delta Y_2}^2 \sigma_{\Delta Y_3}^2] \left. \right\}. \end{aligned}$$

Because the two methods offer quite different results, the use of our formulae may increase the precision of the mechanisms tolerance study and design.

Received November 13, 1982

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APROXIMAȚIE DE ORDINUL AL DOILEA PENTRU MEDIE ȘI DISPERSIE ÎN STUDIUL TOLERANȚELOR LANȚURILOR CINEMATICE

(Rezumat)

Se propun formule aproximative pentru media și dispersia funcțiilor de variabile aleatoare, bazate pe dezvoltarea în serie Taylor și reținerea termenilor de ordinul I și II; formulele sînt utile în studiul probabilistic al toleranțelor lanțurilor cinematice. Aplicarea formulelor respective se exemplifică pe un mecanism bielă-manivelă.

THE GREEN'S FUNCTIONS, FOR SHORT TIME, IN THE GENERALIZED THEORY OF THERMOELASTICITY

I. GENERAL SOLUTION

BY

ELISABETA RUSU

1. Introduction. A.E. Green and K.A. Lindsay [1] have given a generalized theory of thermoelasticity. I. Nistor [2] has obtained a representation of Galerkin type and has derived the solution corresponding to concentrated load in an infinite space by means of the Laplace transform, for small value of time.

Analogous problems in classical theory of linear thermoelasticity have been studied in [3], [4].

In the first part of this paper, we consider the problem of determining the displacements and temperature in an infinite medium under action of any given time dependent body forces and heat sources for the equations of the generalized theory of thermoelasticity [1], [5], by means of Fourier-Laplace transform.

In the second part of paper we will derive the solution for an impulsive body force and heat source acting in a infinite medium, i.e. the fundamental solution.

Analogous problems, in classical theory of linear micropolar thermoelasticity, have been studied in [6].

2. Basic Equations. In this paper we employ a rectangular coordinate system Ox_i ($i=1, 2, 3$) and the usual indicial notation.

The basic equations of the linear theory for an isotropic solid with a center of symmetry at each point are [5]:

$$(2.1) \quad \mu \Delta \mathbf{u} + (\lambda + \mu) \text{grad div } \mathbf{u} - m \text{grad} (\theta + \alpha \dot{\theta}) + \mathbf{b} = \rho \ddot{\mathbf{u}},$$

$$k \Delta \theta - d \dot{\theta} - h \ddot{\theta} - \theta_0 m \text{div } \dot{\mathbf{u}} = -\rho r,$$

where $\mathbf{u}$ — the displacement vector, θ_0 — the reference temperature; θ — the change from the reference temperature; ρ — the density; $\mathbf{b} = \rho \mathbf{F}$, where $\mathbf{F}$ is the body force; r is the externally supplied specific heat; $m, \lambda, \mu, \alpha, d, h, k$, are characteristic constants of the medium; a superposed dot denote material time differentiation and $_{,i}$ denotes partial differentiation with respect to x_i .

We consider the following initial and boundary conditions:

$$\mathbf{u}(x, 0) = 0, \quad \dot{\mathbf{u}}(x, 0) = 0, \quad \theta(x, 0) = 0, \quad \dot{\theta}(x, 0) = 0$$



$$2.2) \quad \lim_{x \rightarrow \infty} u(x, t) = 0, \quad \lim_{x \rightarrow \infty} \theta(x, t) = 0,$$

where x is the position vector.

3. The General Solution of the Basic Equations. To solve these equations we shall first reduce the equations (2.1) to a more simpler form by decomposing the vectors u and b into their potential and solenoidal parts:

$$(3.1) \quad \begin{aligned} u &= \text{grad } \Phi + \text{rot } \Psi, & \text{div } \Psi &= 0; \\ b &= \text{grad } \chi + \text{rot } \nu, & \text{div } \nu &= 0. \end{aligned}$$

Substitution of (3.1) into the basic equations (2.1) yields

$$\begin{cases} \text{grad}[(\lambda + 2\mu)\Delta\Phi - \rho\ddot{\Phi} - m(\ddot{\theta} + \alpha\dot{\theta}) + \chi] + \text{rot}[\mu\Delta\Psi - \rho\ddot{\Psi} + \nu] = 0, \\ k\Delta\theta - d\dot{\theta} - h\ddot{\theta} - \theta_0 m \text{div} \frac{\partial}{\partial t} [\text{grad } \Phi + \text{rot } \Psi] = -\rho r. \end{cases}$$

These equations can be satisfied if we take:

$$(3.2) \quad \begin{cases} \square_1 \Phi = \frac{1}{\lambda + 2\mu} [m(\ddot{\theta} + \alpha\dot{\theta}) - \chi], \\ \square_2 \Psi = -\frac{1}{\mu} \nu, \\ D\theta = \frac{1}{k} \theta_0 m \frac{\partial}{\partial t} \Delta\Phi - \frac{\rho r}{k}, \end{cases}$$

where

$$(3.3) \quad \begin{aligned} \square_1 &= \Delta - \frac{1}{c_1^2} \frac{\partial^2}{\partial t^2}, & c_1^2 &= \frac{\lambda + 2\mu}{\rho}, \\ \square_2 &= \Delta - \frac{1}{c_2^2} \frac{\partial^2}{\partial t^2}, & c_2^2 &= \frac{\mu}{\rho}, \\ D &= \Delta - \frac{d}{k} \frac{\partial}{\partial t} - \frac{h}{k} \frac{\partial^2}{\partial t^2}, & \Delta &= \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_k}. \end{aligned}$$

From (3.2) we find the equations for Φ and θ :

$$(3.4) \quad \begin{cases} D \square_1 - \frac{\theta_0 m^2}{k(\lambda + 2\mu)} \frac{\partial}{\partial t} \left(1 + \alpha \frac{\partial}{\partial t} \right) \Delta \Phi = -\frac{\Delta \chi}{\lambda + 2\mu} - \frac{m \rho}{k(\lambda + 2\mu)} \left(1 + \alpha \frac{\partial}{\partial t} \right) r, \\ D \square_1 - \frac{\theta_0 m^2}{k(\lambda + 2\mu)} \frac{\partial}{\partial t} \left(1 + \alpha \frac{\partial}{\partial t} \right) \Delta \theta = -\frac{m \theta_0}{k(\lambda + 2\mu)} \frac{\partial}{\partial t} \Delta \chi - \frac{\rho}{k} \square_1 r. \end{cases}$$

To solve the waves equations (3.2) let us introduce the Laplace transform $\mathcal{F}^{(1)}(x_1, x_2, x_3, p)$ of the function $\mathcal{F}(x_1, x_2, x_3, t)$ and also introduce the Fourier-Laplace transform $\mathcal{F}^{(2)}(\zeta_1, \zeta_2, \zeta_3, p)$ of the function $\mathcal{F}^{(1)}(x_1, x_2, x_3, p)$.

Application of Laplace-Fourier's transform to the equations (3.4), (3.2) and (3.1) yields the following algebraic equations:

$$(3.5) \quad \begin{cases} \Delta_1 \Phi^{(2)}(\zeta, p) = \frac{\zeta^2 + q + f}{\lambda + 2\mu} \chi^{(2)} - \frac{\rho m}{k(\lambda + 2\mu)} (1 + \alpha p) r^{(2)}, \\ \Delta_1 \theta^{(2)}(\zeta, p) = \frac{\theta_0 m}{k(\lambda + 2\mu)} p \zeta^2 \chi^{(2)} + (\zeta^2 + \beta_1^2) \frac{\rho r^{(2)}}{k}; \end{cases}$$

$$(3.6) \quad (\zeta^2 + \beta_2^2) \Psi_j^{(2)} = \frac{1}{\mu} v_j^{(2)};$$

$$(3.7) \quad \begin{cases} u_i^{(2)}(\zeta, p) = -i \zeta_i \Phi^{(2)} - i \zeta_j \varepsilon_{ijk} \Psi_k^{(2)}, \\ b_i^{(2)}(\zeta, p) = -i \zeta_i \chi^{(2)} - i \zeta_j \varepsilon_{ijk} v_k^{(2)}, \end{cases}$$

where

$$(3.8) \quad \Delta_1 = (\zeta^2 + q + f)(\zeta^2 + \beta_1^2) + \varepsilon q(1 + \alpha p) \zeta^2$$

and

$$(3.9) \quad \begin{cases} q = \frac{d p}{k}, & f = \frac{h p^2}{k}, & \beta_1 = \frac{p}{c_1}, \\ \beta_2 = \frac{p}{c_2}, & \varepsilon = \frac{\theta_0 m^2}{d(\lambda + 2\mu)}, \\ \bar{\zeta}^2 = \zeta_k \zeta_k. \end{cases}$$

From (3.7) along with conditions $\text{div } \Psi = 0$, $\text{div } v = 0$, we find

$$(3.10) \quad \chi^{(2)} = \frac{i}{\zeta^2} \zeta_j b_j^{(2)}, \quad v_j^{(2)} = -\frac{i}{\zeta^2} \zeta_k \varepsilon_{jki} b_i^{(2)}.$$

Now, from (3.6) and (3.10) we get

$$\Psi_j^{(2)} = -\frac{i}{\mu \zeta^2 (\zeta^2 + \beta_2^2)} \zeta_k \varepsilon_{jki} b_i^{(2)}$$

so that

$$(3.11) \quad -i \zeta_j \varepsilon_{ijk} \Psi_k^{(2)} = -\frac{1}{\mu \zeta^2 (\zeta^2 + \beta_2^2)} (\zeta_i \zeta_j b_j^{(2)} - \zeta^2 b_i^{(2)}).$$

From (3.7), (3.5), (3.10) and (3.11) we obtain

$$(3.12) \quad \begin{cases} u_j^{(2)}(\zeta, p) = \frac{1}{\lambda + 2\mu} \left(\frac{\zeta^2 + q + f}{\Delta_1 \zeta^2} \zeta_j \zeta_k b_k^{(2)} + \frac{m}{\Delta_1} i \zeta_j (1 + \alpha p) \frac{\rho r^{(2)}}{k} \right) + \\ + \frac{\zeta^2 b_j^{(2)} - \zeta_j \zeta_k b_k^{(2)}}{\mu \zeta^2 (\zeta^2 + \beta_2^2)}, \\ \theta^{(2)}(\zeta, p) = \frac{\theta_0 m}{k(\lambda + 2\mu)} \frac{p}{\Delta_1} i \zeta_j b_j^{(2)} + \frac{\zeta^2 + \beta_1^2}{\Delta_1} \frac{\rho r^{(2)}}{k}. \end{cases}$$

Thus, the above system give rise to the solution allowing the determination of displacement and temperature field for any given body for-

ces and heat sources applied in an infinite medium by first inverting the Fourier transform and then by inverting the Laplace transform.

In the second part of the paper we seek first a solution of equations of motion when a body force is impulsively applied at the origin. For such body force we may write

$$(3.13) \quad \begin{aligned} b_j &= \delta(x)\delta(y)\delta(z)\delta(t)\delta_{3j}, \\ r &= 0, \end{aligned}$$

if the body force is parallel to the Ox_3 axis; in a similar way we can obtain the solution in the case of impulsive force acting along the axis Ox_1 and along the axis Ox_2 .

After this we seek a solution of equations of motion when a heat source acts in the infinite thermoelastic medium impulsively at origin:

$$(3.14) \quad b_j = 0 \quad (j=1, 2, 3), \quad r = \frac{k}{\rho} \delta(x)\delta(y)\delta(z)\delta(t).$$

The solution can be used as a Green type solution; it is the fundamental solution.

Received September 15, 1983

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FUNCTII GREEN PENTRU TIMP SCURT, ÎN O TEORIE GENERALIZATĂ A TERMOELASTICITĂȚII

I. O soluție generală

(Rezumat)

În partea I-a lucrării se consideră problema determinării deplasărilor și a temperaturii într-un mediu termoelastic infinit, sub acțiunea unor forțe masice și surse date, variabile în raport cu timpul, prin aplicarea transformatei Laplace-Fourier.

În partea a II-a se va deduce soluția pentru forța masică și sursa acționind impulsiv în origine.

A UNIQUENESS THEOREM FOR THE THEORY OF LINEAR THERMOELASTICITY WITH MICROSTRUCTURE

BY

SILVIA GABRIELA CIUMAȘU

1. Introduction. This paper is concerned with the theory of thermoelasticity with microstructure established in [4]. Using some results of [2], [5] we prove a uniqueness theorem in linear thermoelastodynamics for homogeneous solids.

2. Basic Equations. Let Ω be a regular domain in the three-dimensional space occupied by an elastic body with microstructure whose boundary is Γ . We refer the motion of the continuum to a fixed system of rectangular cartesian axes OX_i ($i=1, 2, 3$). We shall employ the usual summations and differentiation conventions.

The basic equations in generalized theory of linear thermoelasticity of a homogeneous with microstructure and centrosymmetric continuum are:

i) *The equations of motions*

$$(2.1) \quad t_{ki,k} + \rho F_i = \rho \ddot{u}_i,$$

$$(2.2) \quad \bar{\sigma}_{ij} + \sigma_{kij,k} + \rho \Phi_{ij} = J_{js} \ddot{\Psi}_{is},$$

$$t_{ij} = s_{ij} + \bar{\sigma}_{ij}, \quad s_{ij} = s_{ji} \quad \text{in } (\Omega \times [0, t_0]),$$

where t_0 is some instant that may be infinite.

ii) *The equation of energy*

$$(2.3) \quad \theta_0 \dot{\rho} S + q_{i,i} = \rho r.$$

iii) *The constitutive equations*

$$s_{ij} = A_{ijkl} \epsilon_{kl} + B_{ijkl} \gamma_{kl} + a_{ij}(\theta + \alpha \dot{\theta}),$$

$$\bar{\sigma}_{ij} = B_{kl} \epsilon_{kl} + C_{ijkl} \gamma_{kl} + c_{ij}(\theta + \alpha \dot{\theta}),$$

$$(2.4) \quad \sigma_{kij} = D_{kijlsm} \chi_{lsm}, \quad q_i = -\theta_c k_{ij} \theta_{,j},$$

$$\rho S = a + d\theta + h\dot{\theta} - a_{ij} \epsilon_{ij} - c_{ij} \gamma_{ij}.$$

iv) *The kinematic relations*

$$(2.5) \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \gamma_{ij} = u_{j,i} - \Psi_{ij}, \quad \kappa_{ijk} = \Psi_{jk,i}.$$

In the above equations, we have used the following notations: u_i — components of the displacement vector; Ψ_{ij} — components of micro-deformation; ε_{ij} , γ_{ij} , κ_{ijk} — kinematic characteristics of the strain; s_{ij} — components of the classical stress tensor; $\bar{\sigma}_{ij}$ — components of the relative stress tensor; σ_{ijk} — components of the double stress tensor; F_i , Φ_{ij} — components of the body forces; q_i — components of the heat conduction vector; S — specific entropy; ρ — density; r — externally applied heat per unit mass; θ_0 — constant reference temperature; J_{js} , A_{ijkl} , B_{ijkl} , C_{ijkl} , a_{ij} , c_{ij} , D_{kijlsm} , d , h , a — characteristic constants of the material. The thermo-elastic coefficients are subject to the symmetry

$$(2.6) \quad \begin{aligned} A_{ijkl} &= A_{klij} = A_{jikl}, & C_{ijkl} &= C_{klij}, \\ B_{ijkl} &= B_{jikl}, & J_{ts} &= J_{st}, \\ D_{kijlsm} &= D_{lsmkij}, & k_{ij} &= k_{ji}. \end{aligned}$$

If the solid has a centre of symmetry at each point, then $b_i = 0$ and the tensors of odd rank must vanish.

From the residual entropy inequality [4] we have the condition

$$(2.7) \quad (\alpha d - h)\dot{\theta}^2 + k_{ij}\theta_i\dot{\theta}_j \geq 0.$$

The density ρ and temperature θ_0 are given constants subject to the conditions

$$(2.8) \quad \rho > 0, \quad \theta_0 > 0,$$

To Eqs. (2.1)–(2.5) we adjoin the boundary conditions

$$(2.9) \quad \begin{aligned} u_i &= \tilde{u}_i \text{ on } \Gamma_1 \times [0, t_0]; \quad \Psi_{ij} = \tilde{\Psi}_{ij} \text{ on } \Gamma_3 \times [0, t_0], \\ t_i &= t_{ij}n_j = \tilde{t}_i \text{ on } \Gamma_2 \times [0, t_0], \\ T_{ij} &= \sigma_{kij}n_k = \tilde{T}_{ij} \text{ on } \Gamma_4 \times [0, t_0], \end{aligned}$$

$$\theta = \tilde{\theta} \text{ on } \Gamma_5 \times [0, t_0], \quad q = q_in_i = \tilde{q}_i \text{ on } \Gamma_6 \times [0, t_0],$$

where Γ_p ($p=1, 2, \dots, 6$) denotes subsets of Σ so that

$$\Gamma_1 \cup \Gamma_2 = \Gamma_3 \cup \Gamma_4 = \Gamma_5 \cup \Gamma_6 = \Gamma, \quad \Gamma_1 \cap \Gamma_2 = \Gamma_3 \cap \Gamma_4 = \Gamma_5 \cap \Gamma_6 = \emptyset$$

and the initial conditions:

$$(2.10) \quad \begin{aligned} u_i(x, 0) &= u_{i0}(x), & \dot{u}_i(x, 0) &= v_{i0}(x), \\ \Psi_{ij}(x, 0) &= \Psi_{ij0}(x), & \dot{\Psi}_{ij}(x, 0) &= v_{ij0}(x), \\ \theta(x, 0) &= T_0(x), & \dot{\theta}(x, 0) &= \dot{T}_0(x), \\ & (i=1, 2, 3; x \in \bar{\Omega}). \end{aligned}$$

In these above relation $\tilde{u}_i$, $\tilde{\Psi}_{ij}$, $\tilde{t}_i$, $\tilde{T}_{ij}$, $\tilde{\theta}$, $\tilde{q}_i$, u_{i0} , Ψ_{ij0} , T_0 , v_{i0} , v_{ij0} , $\dot{T}_0$ are prescribed functions and n_i denote the components unit outward normal to Γ .

3. The Theorem of Uniqueness. Suppose that

$$(3.1) \quad \frac{1}{2} A_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{2} C_{ijkl} \gamma_{ij} \gamma_{kl} + \frac{1}{2} D_{kijlsm} \kappa_{kij} \kappa_{lsm} + B_{ijkl} \gamma_{ij} \varepsilon_{ij} \geq 0,$$

$$J_{js} \dot{\Psi}_{ij} \dot{\Psi}_{is} > 0, \quad \alpha > 0, \quad h > 0, \quad \rho > 0, \quad \theta > 0,$$

then the mixed problem has at most one solution.

The proof rests on the identity

$$(3.2) \quad \int_{\bar{\Gamma}} \left[t_{ji} \dot{u}_i n_j + \sigma_{kij} \dot{\Psi}_{ij} n_k - \frac{q_i}{\theta_0} (\theta + \alpha \dot{\theta}) n_i \right] ds + \int_{\Omega} \left[\rho F_i \dot{u}_i + \rho \Phi_{ij} \dot{\Psi}_{ij} + \right. \\ \left. + \frac{\rho r}{\theta_0} (\theta + \alpha \dot{\theta}) \right] d\tau = \frac{d}{dt} \left\{ \frac{1}{2} \int_{\Omega} \left[\rho \dot{u}_i \dot{u}_i + J_{js} \dot{\Psi}_{is} \dot{\Psi}_{ij} + A_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \right. \right. \\ \left. + C_{ijkl} \gamma_{ij} \gamma_{kl} + D_{kijlsm} \kappa_{kij} \kappa_{lsm} + 2B_{ijkl} \gamma_{kl} \varepsilon_{ij} + \alpha k_{ij} \theta_{,j} \theta_{,i} + \right. \\ \left. + \alpha (\alpha d - h) \left(\frac{\theta}{\alpha} \right)^2 + \frac{h}{\alpha} (\theta + \alpha \dot{\theta})^2 \right] d\tau + \int_{\Omega} \left[(\alpha d - h) \dot{\theta}^2 + k_{ij} \dot{\theta}_{,i} \dot{\theta}_{,j} \right] d\tau \Big\},$$

in which the homogeneous forms of (2.1)–(2.3) are used with $F_i = 0$, $\Phi_{ij} = 0$, $r = 0$.

Suppose that there are two solutions. Then their difference $\{\bar{u}_i, \bar{\Psi}_{ij}, \bar{\theta}\}$ corresponds to null data.

If we denote

$$(3.3) \quad U(t) = \frac{1}{2} \int_{\Omega} \left[\rho \dot{u}_i \dot{u}_i + J_{js} \dot{\Psi}_{is} \dot{\Psi}_{ij} + A_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + C_{ijkl} \gamma_{ij} \gamma_{kl} + \right. \\ \left. + D_{kijlsm} \kappa_{kij} \kappa_{lsm} + 2B_{ijkl} \gamma_{kl} \varepsilon_{ij} + \alpha k_{ij} \theta_{,j} \theta_{,i} + \right. \\ \left. + \alpha (\alpha d - h) \left(\frac{\theta}{\alpha} \right)^2 + \frac{h}{\alpha} (\theta + \alpha \dot{\theta})^2 \right] d\tau,$$

then from (3.2) it follows

$$(3.4) \quad \dot{\bar{U}} = - \int_{\Omega} [(\alpha d - h) \dot{\theta}^2 + k_{ij} \dot{\theta}_{,i} \dot{\theta}_{,j}] d\tau.$$

In view of (2.7) we have

$$(3.5) \quad \dot{\bar{U}} \leq 0, \quad t \in [0, t_0],$$

where $\bar{U}$ is the function U corresponding to the difference of the two solutions.

From (3.4) we obtain

$$(3.6) \quad \bar{U}(t) \leq U(0), \quad t \in [0, t_0].$$

In this case we have

$$(3.7) \quad \bar{U}(0)=0, \quad t \in [0, t_0].$$

Taking into account (3.1) we obtain

$$(3.8) \quad \bar{U}(t)=0, \quad t \in [0, t_0].$$

By using standard arguments from (3.3) and (3.8) it follows $\bar{u}_i=0$, $\bar{\Psi}_{ij}=0$, $\bar{\theta}=0$. Thus the uniqueness theorem is proved.

Received February 4, 1983

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TEOREMĂ DE UNICITATE ÎN TEORIA LINIARĂ A TERMOELASTICITĂȚII CU MICROSTRUCTURĂ

(Rezumat)

Se demonstrează o teoremă de unicitate pentru teoria termoeleasticității generalizate, cu microstructură, stabilită în [4], folosind unele rezultate din [2], [5].

ON SOME BOUNDARY VALUE PROBLEMS IN THE MICROPOLAR THERMOELASTICITY OF STATIONARY VIBRATIONS

BY

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1. Introduction. In the specialized literature there are methods for investigating the boundary value problems of stationary vibrations in the classical theory of elasticity, thermoelasticity and micropolar elasticity by means both of the surface and body potentials and of the theory of singular integral equations; these methods are presented in many papers (see e.g. [6]—[9]) and books (see [1], [2]).

In [3], two fundamental problems of stationary vibrations in micropolar thermoelasticity are reduced to Fredholm singular integral equations by using the potential method elaborated and developed by V. D. Kupradze in [2]. The same method has been used by M. R. Agniasvili [10] who has solved, as well, some concrete problems of this theory.

In this paper the differential equations of linear micropolar thermoelasticity of stationary vibrations for an isotropic, homogeneous and centrosymmetric solid are considered. It is supposed that the solid occupies either a three-dimensional domain $D^+ \subset E_3$, bounded by a closed Ljapunov surface S , or an infinite domain D^- , where $D^- = E_3 \setminus D^+ \cup S$, E_3 being the three-dimensional Euclidian space. For this type of solid, either bounded or unbounded, we define two boundary value problems, called interior or exterior ones. By using the radiation conditions defined in [4], [5], some Green identities proved in this paper and the matrix of the fundamental solutions from [3], we deduce the representation for the regular solution of the differential equations of micropolar thermoelasticity in the case of stationary vibrations. These formulae give a reason for the introduction of two surface micropolar thermoelastic potentials for which some properties are presented. Finally, the two considered problem are reduced to singular integral equations of Fredholm type.

2. Basic Equations. For a homogeneous, isotropic and centrosymmetric body occupying the domain D^+ or D^- , the basic equations of the stationary vibrations in the linear micropolar thermoelasticity, in the absence of the body force, couple-body force and heat source, are [3]

$$(1) \quad \begin{cases} [(\mu + \alpha)\Delta + \rho\omega^2]u + (\lambda + \mu - \alpha) \operatorname{grad} \operatorname{div} u + 2\alpha \operatorname{rot} \varphi - \nu \operatorname{grad} \theta = 0, \\ 2\alpha \operatorname{rot} u + [(\gamma + \varepsilon)\Delta + J\omega^2 - 4\alpha]\varphi + (\beta + \gamma - \varepsilon) \operatorname{grad} \operatorname{div} \varphi = 0, \\ i\omega\eta \operatorname{div} u + (\Delta + i\omega/\kappa)\theta = 0, \end{cases}$$

where u , φ , θ are the amplitudes of the displacement vector, rotation vector and increment of temperature, respectively; λ , μ , α , β , γ , ε are mechanical constants; ν , κ , η are thermal constants; ρ , J , ω are the material density, the rotational inertia and the frequency of vibrations, respectively; $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$, and $i = \sqrt{-1}$.

These equations can be written in a matricial form as

$$(2) \quad B\left(\frac{\partial}{\partial x}, \omega\right) V(x) = 0, \quad x(x_1, x_2, x_3) \in D^+ \quad \text{or} \quad x \in D^-,$$

where B is a differential operator whose matrix of the 7×7 type can easily be read from (1) (see also [3]) and $V(x) = (u(x), \varphi(x), \theta(x))^T$, T being here the transposing symbol of the matrix (u, φ, θ) .

The amplitudes of the stress vector t , the couple-stress vector m and the ratio h/k , where h is the heat flux and k is the Fourier's constant are given, respectively, by

$$(3) \quad \begin{cases} t = 2\mu \frac{\partial u}{\partial n_x} + \lambda n_x \operatorname{div} u + (\mu - \alpha) n_x \times \operatorname{rot} u + 2\alpha n_x \times \varphi, \\ m = 2\gamma \frac{\partial \varphi}{\partial n_x} + \beta n_x \operatorname{div} \varphi + (\gamma - \varepsilon) n_x \times \operatorname{rot} \varphi, \quad \frac{h}{k} = \frac{\partial \theta}{\partial n_x}, \end{cases}$$

where $\partial/\partial n_x$ is the derivative in the direction n_x , n_x being the unitary vector of the normal direction to an element of surface in the point x from the body.

Let us notice that t and m can be also written in the form

$$(4) \quad t = T\left(\frac{\partial}{\partial x}, n_x\right)\left(\frac{u}{\varphi}\right) - \nu \theta n_x, \quad m = M\left(\frac{\partial}{\partial x}, n_x\right) \varphi,$$

where T is a differential matrix of the 3×6 type while M is of the 3×3 type. The components T_{ij} and M_{kl} of T and M , respectively, can be easily found from (3) or from [3].

3. The Boundary Value Problems. In the following we shall call „interior problems“ those referring to D^+ with boundary S and „exterior problems“ when the body occupies the infinite domain D^- with a cavity D^+ , whose boundary is also the Ljapunov closed surface S . We are now defining two interior and also two exterior problems.

Interior Problems. To find $V(x) = (u(x), \varphi(x), \theta(x))^T$ defined on D^+ , of class $C^2(D^+) \cap C^1(D^+)$, $D^+ = D^+ \cup S$, satisfying (2) and one of the boundary conditions

$$(5) \quad \lim_{D^+ \ni x \rightarrow z \in S} \left(u(x), \varphi(x), \frac{\partial \theta}{\partial n_x} \right) = (f(z), g(z), h(z)),$$

$$(6) \quad \lim_{D^+ \ni x \rightarrow z \in S} (t(x), m(x), \theta(x)) = (p_i(z), q_i(z), r_i(z)).$$

Therefore, this means the finding of displacement vector u , the rotation vector φ and the temperature θ , when on S are given either the displacements, the rotations and the heat flux (see (5)), or the stress vector t , the couple-stress m and the temperature (see (6)); n_x is oriented from D^+ to D^- .

These boundary conditions can also be written in the matricial forms

$$(7) \quad \lim_{D^+ \ni x \rightarrow z \in S} Q \left(\frac{\partial}{\partial x}, n_x \right) V(x) = F_i(z), \quad F_i = (f_i, g_i, h_i)^T,$$

$$(8) \quad \lim_{D^+ \ni x \rightarrow z \in S} \mathcal{P} \left(\frac{\partial}{\partial x}, n_x \right) V(x) = G_i(z), \quad G_i = (p_i, q_i, -r_i)^T,$$

where the boundary operators Q and $\mathcal{P}$ are given by

$$(9) \quad Q = \begin{bmatrix} & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial}{\partial n_x} \end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix} T \left(\frac{\partial}{\partial x}, n_x \right) & -\nu n_1 \\ & -\nu n_2 \\ & -\nu n_3 \\ \hline M \left(\frac{\partial}{\partial x}, n_x \right) & 0 \\ & 0 \\ & 0 \\ \hline 0 & 0 & 0 & -1 \end{bmatrix},$$

$E_{6 \times 6}$ being the unity matrix of the 6-th order and $O_{4 \times 3}$ is the nul matrix of the 4×3 type.

Exterior Problems. To find $V(x)$, defined on D^- , of class $C^2(D^-) \cap C^1(D^-)$ satisfying (2), the radiation conditions (see [4]) and one of the boundary conditions

$$(10) \quad \lim_{D^+ \ni x \rightarrow z \in S} Q \left(\frac{\partial}{\partial x}, n_x \right) V(x) = F_e(z), \quad F_e = (f_e, g_e, h_e)^T,$$

$$(11) \quad \lim_{D^+ \ni x \rightarrow z \in S} \mathcal{P} \left(\frac{\partial}{\partial x}, n_x \right) V(x) = G_e(z), \quad G_e = (p_e, q_e, -r_e)^T,$$

where F_e, G_e , like F_i, G_i , are known vector functions of seven components given on S , satisfying the same regularity conditions as in [1].

4. The Green Formulae. In the following we denote by $\tilde{B}(\partial/\partial x, \omega)$ the adjoint operator of the operator $B(\partial/\partial x, \omega)$, in the Lagrange's sense [3], by $\tilde{\mathcal{P}}(\partial/\partial x, n_x)$ an operator obtained from $\mathcal{P}(\partial/\partial x, n_x)$ by changing ν with $i\omega\eta$ in the seventh column, by $\tilde{V}(x)$ a vector function similarly to $V(x)$. Besides, when $x \in D^-$, we shall suppose that $V(x)$ and $\tilde{V}(x)$ satisfy the radiation conditions.

Starting from the Green formulae (4.8) from [3], and (5.3) from [5] and keeping in mind (9), we obtain the following two identities (one for interior and the other for exterior) :

$$(12) \quad \int_{D^+} \left[V(x) \cdot \tilde{B} \left(\frac{\partial}{\partial x}, \omega \right) \tilde{V}(x) - \tilde{V}(x) \cdot B \left(\frac{\partial}{\partial x}, \omega \right) V(x) \right] dx = \\ = \int_S \left[Q \left(\frac{\partial}{\partial x}, n_x \right) V(x) \cdot \tilde{\mathcal{P}} \left(\frac{\partial}{\partial x}, n_x \right) \tilde{V}(x) - \right. \\ \left. - Q \left(\frac{\partial}{\partial x}, n_x \right) \tilde{V}(x) \cdot \mathcal{P} \left(\frac{\partial}{\partial x}, n_x \right) V(x) \right] dS_x,$$

$$(13) \quad \int_{D^-} \left[V \cdot \tilde{B} \tilde{V} - \tilde{V} \cdot B V \right] dx = - \int_S \left[Q V \cdot \tilde{\mathcal{P}} \tilde{V} - Q \tilde{V} \cdot \mathcal{P} V \right] dS_x,$$

where the point means the scalar product in a space with seven dimensions.

5. Integral Representation Formulae. Using the same method as in [3], [5], [9] from (12) and (13) we get

$$(14) \quad l(x) V(x) = \int_S \left(Q \left(\frac{\partial}{\partial y}, n_y \right) \Gamma^T(x, y; \omega, \nu) \right)^T \mathcal{P} \left(\frac{\partial}{\partial y}, n_y \right) V(y) dS_y - \\ - \int_S \left(\tilde{\mathcal{P}} \left(\frac{\partial}{\partial y}, n_y \right) \Gamma^T(x, y; \omega, \nu) \right)^T Q \left(\frac{\partial}{\partial y}, n_y \right) V(y) dS_y,$$

where $l(x)=1$ if $x \in D^+$, $l(x)=1/2$ if $x \in S$, $l(x)=0$ if $x \in D^-$, and

$$(15) \quad \mu(x) V(x) = \int_S \left(\tilde{\mathcal{P}} \left(\frac{\partial}{\partial y}, n_y \right) \Gamma^T(x, y; \omega, \nu) \right)^T Q \left(\frac{\partial}{\partial y}, n_y \right) V(y) dS_y - \\ - \int_S \left(Q \left(\frac{\partial}{\partial y}, n_y \right) \Gamma^T(x, y; \omega, \nu) \right)^T \mathcal{P} \left(\frac{\partial}{\partial y}, n_y \right) V(y) dS_y,$$

where $\mu(x)=1$ if $x \in D^-$, $\mu(x)=1/2$ if $x \in S$, $\mu(x)=0$ if $x \in D^+$, and $\Gamma(x, y; \omega, \nu)$ is the matrix of the fundamental solutions of the operator $B(\partial/\partial x, \omega)$ (see [3]), Γ^T being here the transposed matrix of the matrix Γ .

When $x \in D^+$, from (14), we obtain an integral representation formula for the regular solution in D^+ of Eqs. (1) or (2), while (15) give us, when $x \in D^-$, an integral representation formula for the regular solution of (2) for D^- satisfying the radiation conditions.

6. Surface Potentials. Formulae (14) and (15) give a reason for the introduction of the following two micropolar thermoelastic potentials :

$$(16) \quad Q(x; \varphi) = \int_S \left(Q \left(\frac{\partial}{\partial x}, n_x \right) \Gamma^T(x, y; \omega, \nu) \right)^T \varphi(y) dS_y,$$

$$(17) \quad P(x; \psi) = \int_S \left(\tilde{\mathcal{P}} \left(\frac{\partial}{\partial x}, n_x \right) \Gamma^T(x, y; \omega, \nu) \right)^T \psi(y) dS_y,$$

where φ and ψ are unknown vector functions, called potential densities.

These potentials satisfy (2) in D^+ and in D^- the radiation conditions and the following discontinuity relations:

$$(18) \quad \left[\mathcal{P} \left(\frac{\partial}{\partial z}, n_z \right) Q(z; \varphi) \right]^\pm = \pm \frac{1}{2} \varphi(z) + \\ + \int_S \mathcal{P} \left(\frac{\partial}{\partial z}, n_z \right) \left(Q \left(\frac{\partial}{\partial y}, n_y \right) \Gamma^T(z, y; \omega, \nu) \right)^T \varphi(y) dS_y, \\ (19) \quad \left[Q \left(\frac{\partial}{\partial z}, n_z \right) P(z; \psi) \right]^\pm = \mp \frac{1}{2} \psi(z) + \\ + \int_S Q \left(\frac{\partial}{\partial z}, n_z \right) \left(\tilde{\mathcal{P}} \left(\frac{\partial}{\partial y}, n_y \right) \Gamma^T(z, y; \omega, \nu) \right)^T \psi(y) dS_y,$$

where the symbol „+“ („-“) corresponds to the limit from D^+ (D^-), and the integrals are to be understood in the sense of Cauchy's principal value.

7. Reduction of the Problems to Singular Integral Equations. For the problems, both interior and exterior, (7) and (10) we suppose that the solution has the form

$$(20) \quad V(x) = P(x; \psi),$$

and for both problems (8) and (11), the solution is

$$(21) \quad V(x) = Q(x; \varphi),$$

where the densities of potential are unknown functions defined on S .

In this way $V(x)$ both from (20) and (21) satisfies Eq. (2), but not also the boundary conditions (7)–(8) and (10), (11). From these conditions and the discontinuity relations (18), (19) we obtain the following four singular integral equations for the determination of functions φ and ψ :

$$(22) \quad \mp \frac{1}{2} \psi(z) + \int_S Q \left(\frac{\partial}{\partial z}, n_z \right) \left(\tilde{\mathcal{P}} \left(\frac{\partial}{\partial y}, n_y \right) \Gamma^T(z, y; \omega, \nu) \right)^T \psi(y) dS_y = F_{i,e}(z),$$

$$(23) \quad \pm \frac{1}{2} \varphi(z) + \int_S \tilde{\mathcal{P}} \left(\frac{\partial}{\partial z}, n_z \right) \left(Q \left(\frac{\partial}{\partial y}, n_y \right) \Gamma^T(z, y; \omega, \nu) \right)^T \varphi(y) dS_y = G_{i,e}(z).$$

Using a method similar to that exposed in [1], we can prove that Eqs. (22) and (23) can be regularized; therefore, the Fredholm's theorems are valid for (22) and (23). Finally, we can affirm that the formulated boundary value problems have unique solutions.

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ASUPRA UNOR PROBLEME CU VALORI LA LIMITA ÎN TERMOELASTICITATEA MICROPOLARĂ A VIBRAȚIILOR STAȚIONARE

(Rezumat)

Se consideră două probleme la limită ale ecuațiilor diferențiale ale termoelasticității micropolare a solidelor izotrope, omogene și centrosimetrice, în cazul vibrațiilor staționare. Se reformulează aceste probleme cu ajutorul a doi operatori la limită Q și $\mathcal{P}$. Se demonstrează două identități de tip Green cu ajutorul cărora se dau formule de reprezentare pentru soluția regulată a ecuațiilor de mai sus; aceste formule sugerează introducerea a doi potențiali termoelastici micropolari de suprafață pentru care se prezintă și unele proprietăți. Problemele la limită se reduc la ecuații integrale singulare ce se pot regulariza și cărora li se pot aplica teoremele lui Fredholm în vederea demonstrării existenței soluțiilor problemelor considerate.

WAVE PROPAGATION PROBLEMS IN PLATES

BY

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1. Introduction. The present paper is concerned with some results in the generalized thermoelasticity of plates.

The governing equation for a homogeneous, isotropic, linear thermoelastic plate in a generalized thermoelasticity presented by Müller [1], Green and Lindsay [2], A. E. Green [3], have been given in [4].

We consider a plate with the middle surface in the Ox_1x_2 plane and denote by h his thickness. The equations of motion for a plate in the traverse direction are :

$$(1.1) \quad M_{\sigma\gamma, \sigma\gamma} + B = \rho h \ddot{w},$$

$$(1.2) \quad H - Q_{\sigma, \sigma} - \rho \theta_0 \dot{\eta} - \frac{12k}{h^2} \tau = 0,$$

with

$$B = S_{\sigma, \sigma} + \bar{p}, \quad H = \rho R - \frac{6(\bar{p} + \bar{q})}{h^2},$$

$$S_\gamma = [t_{3\gamma} x_3]_{-\frac{h}{2}}^{\frac{h}{2}} + \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} F_\gamma x_3 dx_3, \quad \bar{p} = [t_{33}]_{-\frac{h}{2}}^{\frac{h}{2}} + \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} F_3 dx_3,$$

$$\bar{p} = q_3 \left(x_1, x_2, -\frac{h}{2} \right), \quad \bar{q} = q_3 \left(x_1, x_2, \frac{h}{2} \right), \quad Q_\sigma = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{12}{h^3} q_\sigma x_3 dx_3,$$

$$(1.3) \quad \eta = \frac{12}{h^3} \int_{-\frac{h}{2}}^{\frac{h}{2}} \tilde{\eta} x_3 dx_3, \quad R = \frac{12}{h^3} \int_{-\frac{h}{2}}^{\frac{h}{2}} r dx_3,$$

$$M_{\sigma_Y} = \int_{-\frac{h}{2}}^{\frac{h}{2}} t_{\sigma_Y} x_3 dx_3, \quad \tau = \frac{12}{h^3} \int_{-\frac{h}{2}}^{\frac{h}{2}} \theta x_3 dx_3,$$

where w is the plate deflection in the direction 3 in terms of t and θ ; t_{ij} — stress tensor; ρ — density; F_i — forces for unit mass; q — the amount of heat generated in a unit volume per unit time; $\tilde{\eta}$ — the entropy; r — externally applied heat per unit mass; θ — temperature; θ_0 — the constant reference temperature; a superposed dot denotes $\partial/\partial t$.

The constitutive equations are:

$$(1.4) \quad M_{\sigma_Y} = -N \left\{ (1-\nu) w_{,\sigma_Y} + \left[\nu \nabla^2 w + \frac{\beta(1+\nu)}{3} (\tau + \alpha \dot{\tau}) \right] \delta_{\sigma_Y} \right\},$$

$$(1.5) \quad \rho \dot{\eta} = d^* \dot{\tau} + h^* \dot{\tau} - \frac{E\beta}{3(1-\nu)} \nabla^2 \dot{w},$$

$$(1.6) \quad Q_\sigma = -k \tau_{,\sigma},$$

where $N = h^3 G / 6(1-\nu)$ and $k, \nu, \beta, G, E, \alpha, d^*, h^*$ are characteristic constants of medium.

Substituting the constitutive Esq. (1.4)–(1.6) in (1.1) and (1.2), we obtain the equations of motion in the form:

$$(1.7) \quad \left(\nabla^4 + \frac{\rho h}{N} \frac{\partial^2}{\partial t^2} \right) w + \frac{\beta(1+\nu)}{3} \nabla^2 \left(1 + \alpha \frac{\partial}{\partial t} \right) \tau = \frac{B}{N},$$

$$(1.8) \quad \frac{E\beta\theta_0}{3k(1-\nu)} \nabla^2 \frac{\partial}{\partial t} w + \left[\nabla^2 - \frac{1}{\alpha} \left(1 + \frac{\varepsilon}{2(1-\nu)} \right) \frac{\partial}{\partial t} - \frac{1}{\alpha} \left(\alpha^* + \frac{\varepsilon\alpha}{2(1-\nu)} \right) \frac{\partial^2}{\partial t^2} + \frac{12}{h^3} \right] \tau = -\frac{H}{k}.$$

Eliminating τ between (1.7) and (1.8), we obtain the equation for the plate deflection

$$(1.9) \quad \begin{aligned} & \nabla^6 w + \frac{\rho h}{N} \nabla^4 \ddot{w} - \frac{1}{k} \left(d^* \theta_0 + \frac{E\beta^2(1+\nu)}{9(1-\nu)} \theta_0 \right) \nabla^4 \dot{w} - \frac{\rho h d^*}{Nk} \theta_0 \ddot{w} - \\ & - \frac{12}{h^2} \left(\nabla^4 w + \frac{\rho h}{N} \ddot{w} \right) - \frac{1}{k} \left(h^* \theta_0 + \frac{E\beta^2 \alpha \theta_0 (1+\nu)}{9(1-\nu)} \right) \nabla^4 \ddot{w} - \\ & - \frac{\rho h^* h \theta_0}{Nk} \ddot{w} = \left(\nabla^2 - \frac{d^* \theta_0}{k} \frac{\partial}{\partial t} - \frac{h^* \theta_0}{k} \frac{\partial^2}{\partial t^2} - \frac{12}{h^2} \right) \frac{B}{N} - \frac{\beta(1+\nu)}{3k} \left(1 + \alpha \frac{\partial}{\partial t} \right) \nabla^2 H, \end{aligned}$$

where we have introduced ε — a dimensionless factor of order 10^{-4} called coupling factor (see [5]). E. I. N. a. n, in [5], has deduced the equation for

the plate deflection in coupled theory of thermoelasticity. If α and α^* are zero, then Eq. (1.9) coincides with that given by E. Inan. Between h^* and α^* there is the relations [3]

$$(1.10) \quad h^* = \frac{\rho c}{\theta_0} \left[\alpha^* + \frac{\varepsilon \alpha}{2(1-\nu)} \right] \quad \text{and} \quad \varepsilon = \frac{2\alpha_1^2 E \theta_0}{\rho c} \frac{1+\nu}{1-2\nu}.$$

2. The Free Vibrations. We consider Eq. (1.9) homogeneous and we assume

$$(2.1) \quad w(x_1, x_2, t) = W(x_1, x_2) e^{i\omega t},$$

$$(2.2) \quad \tau(x_1, x_2, t) = T(x_1, x_2) e^{i\omega t}.$$

The function $W(x_1, x_2)$ verifies the equation

$$(2.3) \quad \left\{ \left(\nabla^4 - \frac{\rho h}{N} \omega^2 \right) \left(\nabla^2 - \frac{i\omega}{\kappa} - \frac{12}{h^2} + \frac{\alpha^* \omega^2}{\kappa} \right) - \frac{i\omega \varepsilon}{x} (1 + i\omega \alpha) \left(\nabla^4 - \frac{\rho h \omega^2}{2N(1-\nu)} \right) \right\} W = 0.$$

For $\varepsilon=0$ Eq. (2.3) takes the form

$$(2.4) \quad \left(\nabla^4 - \frac{\rho h}{N} \omega^2 \right) \left(\nabla^2 - \frac{i\omega^*}{\kappa} - \frac{12}{h^2} + \frac{\alpha^* \omega^{*2}}{\kappa} \right) = 0.$$

The Eq. (2.4) can be separated into two partial differential equations one of 4th order and other of 2nd order.

The algebraic equation of 6th degree associated with Eq. (2.4) is

$$(2.5) \quad \left(\lambda^4 - \frac{\rho h}{N} \omega^{*2} \right) \left(\lambda^2 - \frac{i\omega^*}{\kappa} - \frac{12}{h^2} + \frac{\alpha^* \omega^{*2}}{\kappa} \right) = 0.$$

Here ω^* represents the value of ω for $\varepsilon=0$.

The roots of Eq. (2.5) are

$$(2.6) \quad \lambda_1^{*2} = \sqrt{\frac{\rho h}{N}} \omega^*, \quad \lambda_2^{*2} = -\sqrt{\frac{\rho h}{N}} \omega^*, \quad \lambda_3^{*2} = \frac{i\omega^*}{\kappa} + \frac{12}{h^2} - \frac{\alpha^*}{\kappa} \omega^{*2}.$$

Now, we return to Eq. (2.3), and we associate the algebraic equation of 6th degree

$$(2.7) \quad \left(\lambda^4 - \frac{\rho h}{N} \omega^2 \right) \left(\lambda^2 - \frac{i\omega}{\kappa} - \frac{12}{h^2} + \frac{\alpha^* \omega^2}{\kappa} \right) - \frac{i\omega \varepsilon}{x} (1 + i\omega \alpha) \left(\lambda^4 - \frac{\rho h \omega^2}{2N(1-\nu)} \right) = 0,$$

which has the roots λ_1^2 , λ_2^2 , λ_3^2 , depending on ε . For $\varepsilon=0$ they must coincide with λ_1^{*2} , λ_2^{*2} , and λ_3^{*2} .

Expanding in Maclaurin series with respect to ε , the roots of Eq. (2.7) can be written as

$$(2.8) \quad \lambda_i = \lambda_i|_{\varepsilon=0} + \frac{d\lambda_i}{d\varepsilon}\bigg|_{\varepsilon=0} \varepsilon = \lambda_i^* + (\delta\lambda_i) \varepsilon \quad (i=1, 2, 3)$$

and

$$(2.9) \quad \omega = \omega|_{\varepsilon=0} + \frac{d\omega}{d\varepsilon}\bigg|_{\varepsilon=0} \varepsilon = \omega^* + (\delta\omega) \varepsilon.$$

Let be a plate of sides $2a$ and $2b$ with the sides $x_1 = \pm a$ rigid fixed. In this case Eq. (2.7) can be written in the form

$$(2.10) \quad \prod_{i=1}^3 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \lambda_i^2 \right) W = 0.$$

Because $W(x_1, x_2)$ is zero on sides $x_1 = \pm a$, we take for W the series

$$(2.11) \quad W = \sum_{n=1}^{\infty} Y_n(x_2) \sin \frac{n\pi x_1}{a},$$

where Y_n must satisfy the following equation

$$(2.12) \quad \prod_{i=1}^3 (Y_n'' - \mu_i^2) = 0,$$

where

$$(2.13) \quad \mu_i^2 = \lambda_i^2 + \left(\frac{n\pi}{a} \right)^2 \quad (i=1, 2, 3).$$

Then, for W we have

$$(2.14) \quad W = \sum_{n=1}^{\infty} \sin \frac{n\pi x_1}{a} \sum_{i=1}^3 (C_i \cosh \mu_i x_2 + D_i \sinh \mu_i x_2).$$

It is easy to show that the function T is given by

$$(2.15) \quad T = \sum_{n=1}^{\infty} \sin \frac{n\pi x_1}{a} \sum_{i=1}^3 \frac{A \lambda_i^2}{\lambda_i^2 - \xi^2} (C_i \cosh \mu_i x_2 + D_i \sinh \mu_i x_2),$$

$$\xi^2 = \frac{i\omega}{\kappa} \left(1 + \frac{\varepsilon}{2(1-\nu)} \right) - \frac{\omega^2}{\kappa} \left(\alpha^* + \frac{\varepsilon \alpha}{2(1-\nu)} \right) + \frac{12}{h^2}.$$

For free vibrations of a rectangular plate, the functions W and T have six unknown coefficients, to be determined from the boundary conditions at $x_2 = \pm b$. If those conditions are homogeneous, then the six unknown coefficients must verify an algebraic system of six homogeneous equations. For the existence of a nontrivial solution, the coefficients determinant must be equal to zero. This determinant depends on the roots λ_1^2 , λ_2^2 , λ_3^2 , the frequencies ω and the small quantity ε , that is

$$(2.16) \quad \Delta = \Delta(\lambda_1, \lambda_2, \lambda_3, \omega, \varepsilon) = 0.$$

Expanding the determinant Δ in MacLaurin series with respect to ε , we have

$$(2.17) \quad \Delta = \Delta|_{\varepsilon=0} + \left. \frac{d\Delta}{d\varepsilon} \right|_{\varepsilon=0} \varepsilon = \Delta|_{\varepsilon=0} + (\delta\Delta) \varepsilon,$$

where

$$(2.18) \quad \delta\Delta = \frac{\partial\Delta}{\partial\lambda_1} \delta\lambda_1 + \frac{\partial\Delta}{\partial\lambda_2} \delta\lambda_2 + \frac{\partial\Delta}{\partial\lambda_3} \delta\lambda_3 + \frac{\partial\Delta}{\partial\omega} \delta\omega + \frac{\partial\Delta}{\partial\varepsilon} \varepsilon.$$

For $\varepsilon=0$, $\delta\Delta=0$ and $\Delta|_{\varepsilon=0}=\Delta^*$, whence it results the correction of frequencies

$$(2.19) \quad \delta\omega = - \frac{\left. \frac{\partial\Delta}{\partial\varepsilon} \right|_{\varepsilon=0} + \sum_{i=1}^3 \left. \frac{\partial\Delta}{\partial\lambda_i} \right|_{\varepsilon=0} \delta\lambda_i}{\left. \frac{\partial\Delta}{\partial\omega} \right|_{\varepsilon=0}}.$$

Now, we shall calculate the corrections for λ_1 , λ_2 and λ_3 . The roots λ_1 , λ_2 , λ_3 must verify Eq. (2.7) and we denote by f_i the first member of Eq. (2.7) in which λ is replaced by λ_i :

$$(2.20) \quad f_i = \left(\lambda_i^4 - \frac{\rho h}{N} \right) \left(\lambda_i^2 - \frac{i\omega}{\alpha} - \frac{12}{h^2} + \alpha^* \frac{\omega^2}{\alpha} \right) - \frac{i\omega \varepsilon}{\alpha} (1 + i\omega \alpha) \left(\lambda_i^4 - \frac{\rho h \omega^2}{2N(1-\nu)} \right) = 0.$$

Expanding in series with respect to ε the function f_i we have

$$(2.21) \quad f_i = f_i|_{\varepsilon=0} + \left. \frac{df_i}{d\varepsilon} \right|_{\varepsilon=0} \varepsilon = 0, \quad \delta\lambda_i = - \frac{\left. \frac{\partial f_i}{\partial\omega} \right|_{\varepsilon=0} \delta\omega + \left. \frac{\partial f_i}{\partial\varepsilon} \right|_{\varepsilon=0}}{\left. \frac{\partial f_i}{\partial\lambda_i} \right|_{\varepsilon=0}}.$$

Differentiating Eqs. (2.20) with respect to λ_i , ω and ε and substituting into (2.21) we have

$$\delta\lambda_1 = \frac{\frac{2\rho h\omega^*}{N} (\lambda_1^{*2} - \lambda_3^{*2}) \delta\omega + \frac{i\omega^*}{\alpha} (1 + i\alpha\omega^*) \left(\lambda_1^{*4} - \frac{\rho h}{N} \frac{\omega^{*2}}{1-\nu} \right)}{4\lambda_1^{*3} (\lambda_1^{*2} - \lambda_3^{*2})},$$

$$(2.22) \quad \delta\lambda_2 = \frac{\frac{h\rho\omega^*}{2N\lambda_2^{*3}} \delta\omega + \frac{i\omega^*(1+i\alpha\omega^*)}{\alpha} \frac{\lambda_2^{*4} - \frac{\rho h\omega^{*2}}{2N(1-\nu)}}{4\lambda_2^{*3} (\lambda_1^{*2} - \lambda_3^{*2})}}{1},$$

$$\delta\lambda_3 = \frac{\frac{i-2\alpha^*\omega^*}{2\lambda_3^*} \delta\omega - \frac{i\omega^*(1+i\alpha\omega^*)}{\alpha} \frac{\lambda_3^{*4} - \frac{\rho h\omega^{*2}}{2N(1-\nu)}}{2\lambda_3^* (\lambda_3^{*4} - \lambda_1^{*4})}}{1}.$$

3. The Solution of the Equations of Vibrations. We consider the nonhomogeneous equations of vibrations (1.7) and (1.8). We denote by $W(x_1, x_2, p)$ and $T(x_1, x_2, p)$ the Laplace transform of $w(x_1, x_2, t)$ and $\tau(x_1, x_2, t)$ respectively. Applying Laplace transform to Eq. (1.7) and (1.8) we have

$$(3.1) \quad \nabla^4 W + C_1 \nabla^2 T + C_2 W = \bar{B},$$

$$(3.2) \quad \nabla^2 W + C_4 T + C_5 \nabla^2 T = \bar{H},$$

where

$$\begin{aligned} \bar{B} &= \int_0^\infty \frac{B}{N} e^{-pt} dt + \frac{\rho h}{N} (\dot{W}(x, 0) + pW(x, 0)) + \frac{\alpha\beta(1+\nu)}{3} \nabla^2 T(x, 0), \\ C_1 &= \frac{\beta(1+\nu)}{3} (1-\alpha p), \quad C_2 = \frac{\rho h}{N} p^2, \quad C_4 = \frac{3k(1-\nu)}{E\beta\theta_0 p}, \\ (3.3) \quad C_5 &= \frac{3k(1-\nu)}{E\beta\theta_0 p} \left[-\frac{\rho}{\alpha} \left(1 + \frac{\varepsilon}{2(1-\nu)} \right) - \frac{p^2}{\alpha} \left(\alpha^* + \frac{\varepsilon\alpha}{2(1-\nu)} \right) \frac{12}{h^2} \right], \\ \bar{H} &= \frac{3k(1-\nu)}{E\beta\theta_0 p} \left\{ \int_0^\infty -\frac{\bar{H}}{h} e^{-pt} dt + \nabla^2 W(x, 0) - \frac{1}{\alpha} \left(1 + \frac{\varepsilon}{2(1-\nu)} \right) - \right. \\ &\quad \left. - \frac{1}{\alpha} \left(\alpha^* + \frac{\varepsilon\alpha}{2(1-\nu)} \right) \right\} [\dot{\tau}(x, 0) + p\tau(x, 0)]. \end{aligned}$$

Using the same method with E. Inan [5], we introduce the function

$$(3.4) \quad U = \nabla^2 W + C_1 T$$

and obtain the system of equations

$$(3.5) \quad \nabla^2 W + C_1 T - U = 0,$$

$$(3.6) \quad \nabla^2 W + C_4 T + C_5 \nabla^2 T = \bar{H},$$

$$(3.7) \quad \nabla^2 U + C_2 W = \bar{B}.$$

Multiplying Eq. (3.5) by η , Eq. (3.6) by μ and Eq. (3.7) by ν , where η , μ and ν are unknown functions, and adding, we have

$$(3.8) \quad \nabla^2 [(\eta + \mu) W + \mu C_5 T + \nu U] + [\nu C_2 W + (\eta C_1 + \mu C_4) T - \eta U] = \mu \bar{H} + \nu \bar{B}.$$

We take the unknown functions η , μ , ν so that to have

$$(3.9) \quad M(\eta + \mu) = \nu C_2, \quad M\mu C_5 = \eta C_1 + \mu C_4, \quad M\nu = -\eta,$$

where M is a new unknown function. The system (3.9) is a linear algebraic homogeneous system and the coefficients determinant must be equal to zero. It results that the function M must verify the following equation

$$(3.10) \quad (M^2 + C_2)(MC_5 - C_4) + M^2 C_1 = 0.$$

Let M_1 , M_2 and M_3 be the roots of this equation. To each root M_i corresponds a solution of system (3.9) and Eqs. (3.8) can be written as

$$(3.11) \quad \nabla^2 V_i + M_i V_i = f_i,$$

where

$$(3.12) \quad V_i = (\tau_i + \mu_i) W + \mu_i C_5 T + \lambda_i U,$$

$$(3.13) \quad f_i = \mu_i \bar{H} + \nu_i \bar{B}.$$

For a definite problem, Eq. (3.11) must be solved.

For $i=1, 2, 3$ Eqs. (3.12) constitute a system of three algebraic equations which must be solved with respect to the functions W , T and U .

Received September 15, 1983

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PROBLEME DE PROPAGARE A UNDELOR ÎN PLĂCI

(Rezumat)

Se studiază unele probleme privind propagarea undelor termoelastice în plăci, în cazul teoriei generalizate a termoelasticității, în sensul în care a fost introdusă de I. Müller [1], Green și Lindsay [2]. Pornind de la ecuațiile acestei teorii pentru plăci se studiază vibrațiile staționare și apoi folosind transformata Laplace se cercetează cazul general.

STABILITY OF THE SOLUTIONS IN LINEAR MICROPOLAR VISCOELASTICITY

BY

D. MANOLE

1. Introduction. The aim of this paper is to give sufficient stability conditions for the solutions of linear dynamic micropolar viscoelasticity. We are dealing here with the stability of equilibrium solution and the stability of homogeneous mixed initial boundary value problem for prescribed body force and body couple. Stability of the solution in linear elasticity has been considered in [2]. Recently, this problem was investigated in [3].

2. Preliminary Considerations. Throughout this paper we employ a rectangular coordinate system x_k and the indicial notation. We consider a regular (in the sense of Kellog) region Ω of space occupied by a micropolar viscoelastic material, whose boundary $\partial\Omega$ in the time interval $[0, T]$, $0 < T < \infty$. The basic equations of the linear dynamic theory of micropolar viscoelasticity are:

i) *The equations of motion*

$$(2.1) \quad \begin{aligned} t_{ji,t} + F_i &= \rho(x) \ddot{u}_i, \\ m_{ji,t} + e_{ijk} t_{jk} + M_i &= I_{ij}(x) \ddot{\omega}_j. \end{aligned}$$

ii) *The constitutive equations*

$$(2.2) \quad \begin{aligned} t_{ij}(x, t) &= A_{ijkl}(x, t) \gamma_{kl}(x, t) + B_{ijkl}(x, t) x_{kl}(x, t) + \\ &+ \int_0^t \dot{A}_{ijkl}(x, t, s) \gamma_{kl}(x, s) ds + \int_0^t \dot{B}_{ijkl}(x, t, s) x_{kl}(x, s) ds, \\ m_{ij}(x, t) &= B_{klij}(x, t) \gamma_{kl}(x, t) + C_{ijkl}(x, t) x_{kl}(x, t) + \\ &+ \int_0^t \dot{B}_{klij}(x, t, s) \gamma_{kl}(x, s) ds + \int_0^t \dot{C}_{ijkl}(x, t, s) x_{kl}(x, s) ds. \end{aligned}$$

iii) *The geometrical equations*

$$(2.3) \quad \begin{aligned} \gamma_{ij} &= u_{i,j} - e_{kij} \omega_k, \\ \kappa_{ij} &= \omega_{j,i}. \end{aligned}$$

In these relations, we have used the following notations: t_{ij} — components of the stress tensor; m_{ij} — components of the couple stress tensor; F_i — components of the body force vector; M_i — components of the body couple vector; γ_{ij}, κ_{ij} — components of the micropolar strain tensor; u_i — components of the displacement vector; ω_i — components of the microrotation vector; e_{ijk} — alternating symbol; $\rho, I_{ij}, A_{ijkl}, B_{ijkl}, C_{ijkl}$ — characteristics of the material; the comma denotes partial derivation with respect to the space variables x_k and a dot denotes partial derivation with respect to the time t .

The characteristics of the material A_{ijkl}, C_{ijkl} obey the symmetry relations

$$(2.4) \quad A_{ijkl} = A_{klij}, \quad C_{ijkl} = C_{klij}.$$

To the system of field equations we adjoin the initial conditions

$$(2.5) \quad \begin{aligned} u_i(x, 0) &= a_i(x), & \omega_i(x, 0) &= b_i(x), & x &\in \Omega, \\ \dot{u}_i(x, 0) &= c_i(x), & \dot{\omega}_i(x, 0) &= d_i(x), & x &\in \Omega, \end{aligned}$$

and the homogeneous boundary conditions

$$(2.6) \quad \begin{aligned} u_i(x, t) &= 0, \text{ on } \partial\Omega_u \times [0, T], & \omega_i(x, t) &= 0, \text{ on } \partial\Omega_\omega \times [0, T], \\ t_i(x, t) &\equiv t_{ijn_j} = 0 \text{ on } \partial\Omega_t \times [0, T], & m_i(x, t) &\equiv m_{ijn_j} = 0 \text{ on } \partial\Omega_m \times [0, T], \end{aligned}$$

where a_i, b_i, c_i, d_i are prescribed functions, n_i are components of the unit outward normal to $\partial\Omega$, and $\partial\Omega_u, \partial\Omega_\omega, \partial\Omega_t, \partial\Omega_m$ denote subset of such that

$$\partial\Omega = \partial\Omega_u \cup \partial\Omega_t = \partial\Omega_\omega \cup \partial\Omega_m, \quad \partial\Omega_u \cap \partial\Omega_t = \partial\Omega_\omega \cap \partial\Omega_m = 0.$$

Let $C_0^\infty(\Omega)$ be the set of three-dimensional vector functions with compact support in Ω and components of $C^\infty(\Omega)$. Let H_0, H_+ be Hilbert spaces obtained by completion of $C_0^\infty(\Omega)$ under the norms $\|\cdot\|_0, \|\cdot\|_+$ induced by the inner products

$$(u, v)_{H_0} = \int_{\Omega} (u_i v_i + \omega_i \varphi_i) d\Omega, \quad (u, v)_{H_+} = \int_{\Omega} (u_{i,j} v_{i,j} + \omega_{i,j} \varphi_{i,j}) d\Omega$$

respectively, and let H be the completion of $C_0^\infty(\Omega)$ by means of the norm

$$\|u\|_- = \sup \frac{|(u, v)_{H_0}|}{\|v\|_+} \quad \text{where} \quad u = (u_1, u_2, u_3, \omega_1, \omega_2, \omega_3).$$

Definition [1]. A generalized solution of the mixed initial boundary problem (2.1), (2.4), (2.6) on $[0, T]$ is a function $u(x, t) \in L^2([0, T]; H_+)$ with $\dot{u}(x, t) \in L^2([0, T]; H_0)$, $\ddot{u}(x, t) \in L^2([0, T]; H_-)$ which satisfies the system (2.1) almost everywhere on $[0, T]$ and the initial conditions.

If the conditions

a) the elasticities $A_{ijkl}(x, t)$, $B_{ijkl}(x, t)$, $C_{ijkl}(x, t)$ and $\dot{A}_{ijkl}(x, t)$, $\dot{B}_{ijkl}(x, t)$, $\dot{C}_{ijkl}(x, t)$ are Lebesgue measurable functions essentially bounded on Ω for any fixed $t \in [0, T]$;

b) the relation moduli $\dot{A}_{ijkl}(x, t, s)$, $\dot{B}_{ijkl}(x, t, s)$, $\dot{C}_{ijkl}(x, t, s)$ and $\frac{\partial}{\partial s} \dot{A}_{ijkl}(x, t, s)$, $\frac{\partial}{\partial s} \dot{B}_{ijkl}(x, t, s)$, $\frac{\partial}{\partial s} \dot{C}_{ijkl}(x, t, s)$ are Lebesgue measurable functions essentially bounded on Ω for fixed $(t, s) \in [0, T] \times [0, T]$;

c) there exists the constant $\varepsilon > 0$ such that

$$(2.7) \quad \int_{\Omega} [A_{ijkl}(x, t) \gamma_{ij}(x, t) \gamma_{kl}(x, t) + 2B_{ijkl}(x, t) \gamma_{ij}(x, t) \kappa_{kl}(x, t) + C_{ijkl}(x, t) \kappa_{ij}(x, t) \kappa_{kl}(x, t)] d\Omega > \alpha \|u(x, t)\|_+^2$$

for every $u(x, t) \in H_+$, uniformly with respect to $t \in [0, T]$;

d) there exists $\rho_0 > 0$ such that

$$(2.8) \quad \operatorname{ess\,inf}_{x \in \Omega} \rho(x) \geq \rho_0;$$

e) there exists $\gamma > 0$ such that

$$(2.9) \quad I_{ij}(x) \xi_i \xi_j \geq \gamma \xi_k \xi_k \quad \text{and} \quad I_{ij}(x) = I_{ji}(x), \quad \forall \xi(\xi_i);$$

f) the body force $F(x, t) \in L^2([0, T]; H_0)$, the body couple $M(x, t) \in L^2([0, T]; H_0)$;

g) $a \in H_+$, $b \in H_0$, $a = (a_1, a_2, a_3, b_1, b_2, b_3)$, $b = (c_1, c_2, c_3, d_1, d_2, d_3)$ are satisfied, then the problem (2.1), (2.4), (2.6) has a generalized solution [1]. We introduce the notations

$$(2.10) \quad E(t) = \frac{1}{2} \int_{\Omega} (\rho \dot{u}_i^2 + I_{ij} \dot{\omega}_i \dot{\omega}_j + A_{ijkl} \gamma_{ij} \gamma_{kl} + 2B_{ijkl} \gamma_{ij} \kappa_{kl} + C_{ijkl} \kappa_{ij} \kappa_{kl}) d\Omega,$$

$$(2.11) \quad t_{ij}^{(V)} = \int_0^t (\dot{A}_{ijkl} \gamma_{kl} + \dot{B}_{ijkl} \kappa_{kl}) ds, \quad m_{ij}^{(V)} = \int_0^t (\dot{B}_{kl ij} \gamma_{kl} + \dot{C}_{ijkl} \kappa_{kl}) ds,$$

$$(2.12) \quad P(t) = \int_{\Omega} (F_i \dot{u}_i + M_i \dot{\omega}_i) d\Omega;$$

we suppose

$$(2.13) \quad - \int_{\Omega} (\dot{A}_{ijkl} \gamma_{ij} \gamma_{kl} + 2\dot{B}_{ijkl} \gamma_{ij} \kappa_{kl} + \dot{C}_{ijkl} \kappa_{ij} \kappa_{kl}) d\Omega > \delta \|u\|_+^2, \quad \delta > 0$$

for every $u \in H_+$, uniformly with respect to $t \in [0, T]$ and

$$(2.14) \quad \int_{\Omega} (t_{ij}^{(V)} \dot{\gamma}_{ij} + m_{ij}^{(V)} \dot{\kappa}_{ij}) d\Omega \geq 0, \quad t \in [0, T].$$

3. Stability of Equilibrium. In the above conditions we shall extend some results of [3] concerning the stability of the viscoelasticity solutions to the case of micropolar viscoelasticity.

Theorem 3.1 [3]. *The equilibrium solution of dynamical micropolar viscoelasticity is uniformly Ljapunov stable with respect to the measures*

$$(3.1) \quad \mu(u) = \|u(x, t)\|_0^2 + \|u(x, t)\|_+^2, \quad \mu_0(u) = E(0),$$

or in respect to the measures

$$(3.2) \quad \mu(u) = \|\dot{u}(x, t)\|_0^2 + \|u(x, t)\|_+^2, \quad \mu_0(u) = \|\dot{u}(x, 0)\|_0^2 + \|u(x, t)\|_+^2.$$

Proof. Multiplying (2.1)₁ by $\dot{u}_i(x, t)$ and (2.1)₂ by $\dot{\omega}_i(x, t)$, adding, integrating over $[0, t] \times \Omega$, applying the divergence theorem and taking into account (2.6) it results

$$(3.3) \quad \int_{\Omega} (\rho \ddot{u}_i \dot{u}_i + I_{ij} \ddot{\omega}_i \dot{\omega}_j + t_{ij} \dot{\gamma}_{ij} + m_{ij} \dot{\chi}_{ij}) d\Omega = \int_{\Omega} (F_i \dot{u}_i + M_i \dot{\omega}_i) d\Omega.$$

From (3.3) on the basis of (2.2), we obtain

$$(3.4) \quad \begin{aligned} & \int_{\Omega} (\rho \ddot{u}_i \dot{u}_i + I_{ij} \ddot{\omega}_i \dot{\omega}_j + A_{ijkl} \dot{\gamma}_{kl} \dot{\gamma}_{ij} + \dot{B}_{ijkl} \dot{\chi}_{kl} \dot{\gamma}_{ij} + B_{kl ij} \dot{\gamma}_{kl} \dot{\chi}_{ij} + \\ & + C_{ijkl} \dot{\chi}_{kl} \dot{\chi}_{ij}) d\Omega + \iint_{\Omega_0} (\dot{A}_{ijkl} \dot{\gamma}_{kl} \dot{\gamma}_{ij} + \dot{B}_{ijkl} \dot{\chi}_{kl} \dot{\gamma}_{ij} + \dot{B}_{kl ij} \dot{\gamma}_{kl} \dot{\chi}_{ij} + \\ & + \dot{C}_{ijkl} \dot{\chi}_{kl} \dot{\chi}_{ij}) ds d\Omega = \int_{\Omega} (F_i \dot{u}_i + M_i \dot{\omega}_i) d\Omega. \end{aligned}$$

Using the relations (2.10), (2.11), (2.12) from (3.4) we have

$$(3.5) \quad \begin{aligned} E(t) = & \frac{1}{2} \int_{\Omega} (\dot{A}_{ijkl} \dot{\gamma}_{ij} \dot{\gamma}_{kl} + 2 \dot{B}_{ijkl} \dot{\gamma}_{ij} \dot{\chi}_{kl} + \dot{C}_{ijkl} \dot{\chi}_{ij} \dot{\chi}_{kl}) - \\ & - \int_{\Omega} (\dot{t}_{ij}^{(V)} \dot{\gamma}_{ij} + \dot{m}_{ij}^{(V)} \dot{\chi}_{ij}) d\Omega + P(t). \end{aligned}$$

In equilibrium case $F_i(x, t) = 0$, $M_i(x, t) = 0$. In this case $P(t) = 0$ and by (3.5), (2.13), (2.14) for $t > 0$ we have

$$(3.6) \quad E(t) \leq E(0).$$

Using the conditions c)–e) of Sect. 2 we get

$$(3.7) \quad \rho_0' \|u(x, t)\|_0^2 + \alpha \|u(x, t)\|_+^2 \leq 2E(t),$$

where $\rho_0' = \min(\rho_0, \gamma)$ and thus from (3.6) we have

$$(3.8) \quad \|\dot{u}(x, t)\|_0^2 + \|u(x, t)\|_+^2 \leq \frac{2}{m} E(0),$$

where $m = \min(\rho_0', \alpha)$.

If we introduce the notations

$$(3.9) \quad \rho_0'' = \max(\rho, E), \quad D = \operatorname{ess\,sup}_{x \in \Omega} \sqrt{\operatorname{tr}(F^T F)}, \quad E = \operatorname{ess\,sup}_{x \in \Omega} \sqrt{\operatorname{tr}(I^T I)},$$

$$M = \max(\rho_0'', D), \quad F = \begin{bmatrix} A & B \\ B & C \end{bmatrix}_{18 \times 18};$$

I, A, B, C is the matrix of components $I_{ij}, A_{ijkl}(x, 0), B_{ijkl}(x, 0), C_{ijkl}(x, 0)$. We obtain

$$(3.10) \quad \begin{aligned} 2E(0) &= \int_{\Omega} (\rho \dot{u}_i^2(x, 0) + I_{ij}(x) \omega_i(x, 0) \dot{\omega}_j(x, 0) + A_{ijkl}(x, 0) \gamma_{ij}(x, 0) \gamma_{kl}(x, 0) + \\ &+ 2B_{ijkl}(x, 0) \gamma_{ij}(x, 0) \kappa_{kl}(x, 0) + C_{ijkl}(x, 0) \kappa_{ij}(x, 0) \kappa_{kl}(x, 0)) d\Omega \leq \\ &\leq \int_{\Omega} [\rho_0'' (\dot{u}_i^2(x, 0) + \dot{\omega}_i^2(x, 0)) + D(\gamma_{ij}^2(x, 0) + \kappa_{ij}^2(x, 0))] d\Omega \leq \\ &\leq M \int_{\Omega} [\dot{u}_i^2(x, 0) + \dot{\omega}_i^2(x, 0) + \gamma_{ij}^2(x, 0) + \kappa_{ij}^2(x, 0)] d\Omega \leq \\ &\leq M_1 (\|u(x, 0)\|_0^2 + \|u(x, 0)\|_+^2), \quad M_1 > 0. \end{aligned}$$

Now, from (3.8) and (3.10) we obtain the main inequality

$$(3.11) \quad \|\dot{u}(x, t)\|_0^2 + \|u(x, t)\|_+^2 \leq \frac{M_1}{m} (\|\dot{u}(x, 0)\|_0^2 + \|u(x, 0)\|_+^2).$$

This concludes the proof of the theorem.

Corollary 3.1. By using Friedrichs's inequality [4, p. 41]

$$(3.12) \quad \int_{\Omega} \sum_{i=1}^6 u_i^2 d\Omega \leq c \left\{ \int_{\Omega} \sum_{i,k=1}^6 (u_{i,k})^2 d\Omega + \int_{\partial\Omega} u^2 d\sigma \right\}; \quad c > 0$$

we have

$$(3.13) \quad \|u(x, t)\|_0 \leq c \|u(x, t)\|_+.$$

From (3.11) and (3.13) we have

$$(3.14) \quad \|\dot{u}(x, t)\|_0^2 + \|u(x, t)\|_0^2 \leq \frac{M_1}{mm_1} (\|\dot{u}(x, 0)\|_0^2 + \|u(x, 0)\|_0^2)$$

with $m_1 = \min\{1, c\}$. That is the equilibrium solution is stable with respect to the measures

$$(3.15) \quad \mu(u) = \|\dot{u}(x, t)\|_0^2 + \|u(x, t)\|_0^2, \quad \mu_0(u) = E(0),$$

or in respect to the measures

$$(3.16) \quad \mu(u) = \|\dot{u}(x, t)\|_0^2 + \|u(x, t)\|_0^2, \quad \mu_0(u) = \|\dot{u}(x, 0)\|_0^2 + \|u(x, 0)\|_+^2.$$

Theorem 4.2. *The equilibrium solution of linear dynamic micropolar viscoelasticity is uniformly Ljapunov stable with respect to the measures*

$$(3.17) \quad \mu(u) = \int_0^t \|u(x, s)\|_+^2 ds, \quad \mu_0(u) = E(0)$$

or in respect to the measures

$$(3.18) \quad \mu(u) = \int_0^t \|u(x, s)\|_0^2 ds, \quad \mu_0(u) = E(0).$$

Proof. Taking into account the conditions (3.5), (2.13), (2.14), (2.7), (2.8) and (2.9) we have

$$(3.19) \quad m(\|\dot{u}(x, t)\|_0^2 + \|u(x, t)\|_+^2) + \delta \int_0^t \|u(x, s)\|_+^2 ds \leq 2E(0).$$

From (3.19) it follows

$$(3.20) \quad m\|u(x, t)\|_+^2 + \delta \int_0^t \|u(x, s)\|_+^2 ds \leq 2E(0).$$

If we use the Gronwall inequality [6, p. 53] in

$$(3.21) \quad \eta(t) \leq f_1 \int_0^t \eta(s) ds + f_2$$

we have

$$(3.22) \quad \eta(t) \leq f_2 e^{f_1 t}$$

and from (3.20) we get

$$(3.23) \quad \|u(x, s)\|_+^2 \leq \frac{2E(0)}{m} e^{-\frac{\delta}{m}s}.$$

Integrating (3.23) over $[0, t]$ we have the inequality

$$(3.24) \quad \int_0^t \|u(x, s)\|_+^2 ds \leq \frac{2E(0)}{\delta} (1 - e^{-\frac{\delta}{m}t}) \leq \frac{2}{\delta} E(0)$$

and by (3.13) we get

$$(3.25) \quad \int_0^t \|u(x, s)\|_0^2 ds \leq \frac{2c}{\delta} E(0) (1 - e^{-\frac{\delta}{m}t}) \leq \frac{2c}{\delta} E(0).$$

This concludes the proof of the theorem.

Taking $t \rightarrow \infty$ in (3.24) and (3.25), we obtain

$$(3.26) \quad \int_0^{\infty} \|u(x, s)\|_+ ds \leq \frac{2}{\delta} E(0),$$

$$(3.27) \quad \int_0^{\infty} \|u(x, s)\|_0^2 ds \leq \frac{2c}{\delta} E(0).$$

4. Prescribed Body Force, Body Couple and Homogeneous Initial Data. If $F(x, t) \neq 0$, $M(x, t) \neq 0$, $a(x) = b(x) = 0$ from (3.5), (2.13), (2.14) we obtain

$$(4.1) \quad E(t) \leq \iint_{\Omega} (F_i(x, s) \dot{u}_i(x, s) + M_i(x, s) \dot{\omega}_i(x, s)) d\Omega \, ds,$$

(see [3], p. 96).

Received February 25, 1981

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STABILITATEA SOLUȚIILOR ÎN VISCOELASTICITATEA MICROPOLARĂ LINIARĂ

(Rezumat)

Se stabilesc condiții suficiente de stabilitate a soluțiilor din viscoelasticitatea micropolară liniară. După ce stabilitatea soluțiilor din viscoelasticitatea micropolară este redusă la o problemă analogă din viscoelasticitatea liniară, se procedează ca în [3].

WEAK SOLUTIONS IN GENERALIZED THEORY OF SHELLS
WITH NON-UNIFORM THICKNESSBY
GH. CHIORESCU

1. Introduction. The existence of the weak solutions for Cosserat surfaces (called also generalized shells), has been treated by others authors in the past few years too. So, N. Coutris [1] has considered the Naghdi's membrane theory and D. Cesari [2] deals with the isotropic generalized shells with uniform thickness. In the sequel we are concerning in proving the existence of the weak solution for anisotropic generalized shells with non-uniform thickness.

2. Basic Equations of the Theory. A Cosserat surface consists of a surface with a vector (called director) attached to every point of it. Let the particle of the material surface of this body be identified with convected coordinates θ^α , $\alpha=1, 2$. We suppose θ^α from an open bounded set $\Omega \subset R^2$, and we assume the boundary $\partial\Omega$ locally Lipschitzian ([3], [4]).

Let the reference surface of a generalized shell be referred to by $\mathcal{S}$, let $\mathbf{R}$ be the position vector of $\mathcal{S}$ and $\mathbf{D}$ the reference value of the director. Further, let the surface occupied by the material surface in the present configuration at time t be referred to by $\mathcal{I}$, let $\mathbf{r}$ be the position vector of $\mathcal{I}$ and $\mathbf{d}$ the director at $\mathbf{r}$. Then, the motion of a Cosserat surface is defined by

$$(2.1) \quad \mathbf{r} = \mathbf{r}(\theta^\alpha, t), \quad \mathbf{d} = \mathbf{d}(\theta^\alpha, t), \quad [\mathbf{a}_1, \mathbf{a}_2, \mathbf{d}] > 0,$$

where the condition $(1.1)_3$ ensures that $\mathbf{d}$ is not tangent to the surface $\mathcal{I}$. In the reference configuration, the initial position $\mathbf{R}$ of the surface $\mathcal{S}$ and the initial director $\mathbf{D}$ at $\mathbf{R}$ are given by

$$(2.2) \quad \mathbf{R} = \mathbf{R}(\theta^\alpha) = \mathbf{r}(\theta^\alpha, 0), \quad \mathbf{D} = \mathbf{D}(\theta^\alpha) = \mathbf{d}(\theta^\alpha, 0).$$

We summarise now the basic equations of the Naghdi's theory of the Cosserat surface [5].

i) Geometrical equations

$$(2.3) \quad \mathbf{r} = \mathbf{R} + \mathbf{u}, \quad \mathbf{d} = \mathbf{D} + \delta, \quad \mathbf{D} = D\mathbf{A}_3, \quad D = D(\theta^1, \theta^2),$$

$$(2.4) \quad \begin{aligned} e_{\alpha\gamma} &= \frac{1}{2} (u_{\alpha/\gamma} + u_{\gamma/\alpha}) - B_{\alpha\gamma} u_3, \\ x_{\gamma\alpha} &= D_{,\alpha} B_{\gamma}^{\nu} u_{\nu} + D B_{\alpha}^{\beta} B_{\gamma\beta} u_3 - B_{\alpha\gamma} \delta_3 - D B_{\alpha}^{\beta} u_{\beta/\gamma} + D_{,\alpha} u_{3,\gamma} + \delta_{\gamma/\alpha}, \\ s_{\alpha} &= D^2 B_{\alpha}^{\nu} B_{\nu}^{\delta} u_{\delta} + D B_{\alpha}^{\nu} \delta_{\nu} + D^2 B_{\alpha}^{\nu} u_{3,\nu} + D \delta_{3,\alpha} + D_{,\alpha} \delta_3, \\ \gamma_{\alpha} &= D B_{\alpha}^{\nu} u_{\nu} + \delta_{\alpha} + D u_{3,\alpha}, \quad s = 2D \delta_3, \end{aligned}$$

where $\mathbf{u}$ and δ are the infinitesimal displacement and the infinitesimal director displacement respectively, $D=D(\theta^1, \theta^2)$ is the magnitude of $\mathbf{D}$, $e_{\alpha\gamma}$, $\kappa_{\gamma\alpha}$, s_α , γ_α , s , are the deformation measures, $\mathbf{A}_i$ are the base vectors; $\mathbf{A}_\alpha=\partial\mathbf{R}/\partial\theta^\alpha$, $\mathbf{A}_3=(\mathbf{A}_1\times\mathbf{A}_2)/|\mathbf{A}_1\times\mathbf{A}_2|$, $B_{\alpha\gamma}$ are the components of the second fundamental form of $\mathcal{S}$, a vertical bar stands for covariant differentiation with respect to $A_{\alpha\gamma}$ and a comma denotes partial differentiation with respect to the surface coordinates θ^α , $\alpha=1, 2$.

ii) *Constitutive equations*

$$(2.5) \quad \begin{aligned} N^{\alpha\beta} + DM^{\gamma\alpha} B_{\gamma}^{\beta} &= \rho_0 (\partial\Psi/\partial e_{\alpha\beta}), \quad m^\alpha = \rho_0 (\partial\Psi/\partial \gamma_\alpha), \\ m^3 &= \rho_0 (2D(\partial\Psi/\partial s) + D_{,\alpha}(\partial\Psi/\partial s_\alpha)), \quad M^{\alpha\beta} = \rho_0 (\partial\Psi/\partial \kappa_{\beta\alpha}), \\ M^{\alpha 3} &= \rho_0 D(\partial\Psi/\partial s_\alpha), \quad N^{\alpha 3} - m^\alpha D - M^{\gamma 3} DB_{\gamma}^{\alpha} - M^{\gamma\alpha} D_{,\gamma} = 0, \\ &\alpha = 1, 2; \quad i = 1, 2, 3, \end{aligned}$$

where $N^{\alpha i}$, $M^{\alpha i}$, m^i are the contact force, the contact director couple and the intrinsic director couple respectively, ρ_0 is the mass density in the reference state, and Ψ is Helmholtz's free energy function.

iii) *Equilibrium equations*

$$(2.6) \quad \begin{aligned} N_{/\alpha}^{\alpha\beta} - B_{\alpha}^{\beta} N^{\alpha 3} + \rho_0 F^{\beta} &= 0, \quad N_{/\alpha}^{\alpha 3} + B_{\alpha\beta} N^{\alpha\beta} + \rho_0 F^3 = 0, \\ M_{/\alpha}^{\alpha\beta} - B_{\alpha}^{\beta} M^{\alpha 3} + \rho_0 L &= m^{\beta}, \quad M_{/\alpha}^{\alpha 3} + B_{\alpha\beta} M^{\alpha\beta} + \rho_0 L^3 = m^3, \end{aligned}$$

where $\mathbf{F}$ is assigned force vector and $\mathbf{L}$ is assigned director couple.

For an anisotropic Cosserat surface we express Ψ as a quadratic function of $e_{\alpha\beta}$, $\kappa_{\alpha\beta}$, s_α , γ_α , and s . Such that we have [5]

$$(2.7) \quad \begin{aligned} \rho_0 \Psi &= {}_1 C^{\alpha\beta\gamma\delta} e_{\alpha\beta} \gamma_{\gamma\delta} + {}_2 C^{\alpha\beta\gamma\delta} \kappa_{\alpha\beta} \kappa_{\gamma\delta} + {}_3 C^{\alpha\beta\gamma\delta} e_{\alpha\beta} \kappa_{\gamma\delta} + {}_1 C^{\alpha\beta\gamma} s_\alpha \kappa_{\beta\gamma} + \\ &+ {}_2 C^{\alpha\beta\gamma} e_{\alpha\beta} \gamma_{\gamma} + {}_3 C^{\alpha\beta\gamma} e_{\alpha\beta} s_{\gamma} + {}_4 C^{\alpha\beta\gamma} \gamma_{\alpha} \kappa_{\beta\gamma} + {}_1 C^{\alpha\beta} \gamma_{\alpha} \gamma_{\beta} + {}_2 C^{\alpha\beta} s_{\alpha} s_{\beta} + \\ &+ {}_3 C^{\alpha\beta} \gamma_{\alpha} s_{\beta} + {}_4 C^{\alpha\beta} e_{\alpha\beta} s + {}_5 C^{\alpha\beta} \kappa_{\alpha\beta} s + {}_1 C^{\alpha} \gamma_{\alpha} s + {}_2 C^{\alpha} s_{\alpha} s + C(s)^2, \end{aligned}$$

where the coefficients satisfy the symmetry conditions

$$(2.8) \quad \begin{aligned} {}_1 C^{\alpha\beta\gamma\delta} &= {}_1 C^{\beta\alpha\gamma\delta} = {}_1 C^{\alpha\beta\delta\gamma} = {}_1 C^{\gamma\delta\alpha\beta}, \quad {}_2 C^{\alpha\beta\gamma\delta} = {}_2 C^{\gamma\delta\alpha\beta}, \\ {}_3 C^{\alpha\beta\gamma\delta} &= {}_3 C^{\beta\alpha\gamma\delta}, \quad {}_2 C^{\alpha\beta\gamma} = {}_2 C^{\beta\alpha\gamma}, \quad {}_3 C^{\alpha\beta\gamma} = {}_3 C^{\beta\alpha\gamma}, \quad {}_4 C^{\alpha\beta\gamma} = {}_4 C^{\beta\alpha\gamma}, \\ {}_1 C^{\alpha\beta} &= {}_1 C^{\beta\alpha}, \quad {}_2 C^{\alpha\beta} = {}_2 C^{\beta\alpha}, \quad {}_3 C^{\alpha\beta} = {}_3 C^{\beta\alpha}, \quad {}_4 C^{\alpha\beta} = {}_4 C^{\beta\alpha}. \end{aligned}$$

Let $\mathbf{h}$ be the following unknown displacement vector:

$$\mathbf{h} = (u_1, u_2, u_3, \delta_1, \delta_2, \delta_3).$$

3. Weak Formulation of a Boundary Value Problem. Let $\mathcal{L}^2(\mathcal{S})$ denotes the closure of the functions of class $C^\infty(\mathcal{S})$ with respect to the norm

$$(3.1) \quad \|\mathbf{h}\|_0^2 = \int_{\mathcal{S}} \left(\sum_{i=1}^6 h_i^2 \right) d\Sigma,$$

where $d\Sigma = A^{1/2} d\theta^1 d\theta^2$, $A = \det A_{\alpha\beta}$.

Also, we consider the real Sobolev space $\mathcal{H}^1(\mathcal{S})$ obtained from $C^\infty(\mathcal{S})$ by closure with respect to the norm

$$(3.2) \quad \|\mathbf{h}\|_1^2 = \int_{\mathcal{S}} \left(\sum_1^6 h_i^2 \right) d\Sigma + \int_{\mathcal{S}} \left(\sum_{\alpha, i=1}^{6,2} h_{i,\alpha}^2 \right) d\Sigma.$$

The spaces $\mathcal{H}^1(\mathcal{S})$ and $H^1(\Omega)$ (the ordinary Sobolev space) are isomorphic [2].

With the differential operator (2.6) we associate the following sesquilinear form

$$(3.3) \quad \begin{aligned} A(\mathbf{h}, \mathbf{h}') = & \int_{\mathcal{S}} [{}_1C^{\alpha\beta\gamma\delta} e_{\alpha\beta} e'_{\gamma\delta} + {}_2C^{\alpha\beta\gamma\delta} \kappa_{\alpha\beta} \kappa'_{\gamma\delta} + \frac{1}{2} {}_3C^{\alpha\beta\gamma\delta} (e'_{\alpha\beta} \kappa_{\gamma\delta} + e_{\alpha\beta} \kappa'_{\gamma\delta}) + \\ & + \frac{1}{2} {}_1C^{\alpha\beta\gamma} (s'_{\alpha\beta\gamma} + s_{\alpha\beta\gamma}) + \frac{1}{2} {}_2C^{\alpha\beta\gamma} (e'_{\alpha\beta} \gamma_{\gamma} + e_{\alpha\beta} \gamma'_{\gamma}) + \frac{1}{2} {}_3C^{\alpha\beta\gamma} (e'_{\alpha\beta} s_{\gamma} + e_{\alpha\beta} s'_{\gamma}) + \\ & + \frac{1}{2} {}_4C^{\alpha\beta\gamma} (\gamma'_{\alpha\beta\gamma} + \gamma_{\alpha\beta\gamma}) + {}_1C^{\alpha\beta} \gamma'_{\alpha} \gamma_{\beta} + {}_2C^{\alpha\beta} s'_{\alpha} s_{\beta} + \frac{1}{2} {}_3C^{\alpha\beta} (\gamma'_{\alpha} s_{\beta} + \gamma_{\alpha} s'_{\beta}) + \\ & + \frac{1}{2} {}_4C^{\alpha\beta} (e'_{\alpha\beta} s + e_{\alpha\beta} s') + \frac{1}{2} {}_5C^{\alpha\beta} (\kappa'_{\alpha\beta} s + \kappa_{\alpha\beta} s') + \frac{1}{2} {}_1C^{\alpha} (\gamma'_{\alpha} s + \gamma_{\alpha} s') + \\ & + \frac{1}{2} {}_3C^{\alpha} (s'_{\alpha} s + s_{\alpha} s') + {}_0C s s'] d\Sigma. \end{aligned}$$

Let $\mathbf{f}$ be the vector $\mathbf{f} = (\mathbf{F}, \mathbf{L})$ and let us suppose that $\mathbf{f} \in \mathcal{L}_2(\mathcal{S})$. With the form (3.3) we can define a class of boundary value problems. We shall consider only the zero displacement boundary value problem (in sense of the trace). So that we have to find a vector $\mathbf{h} \in \mathbf{V} = \mathcal{H}_0^1(\mathcal{S})$ (the closure of $C^\infty(\mathcal{S})$ in respect with the norm (2.3)) such that

$$(3.4) \quad A(\mathbf{h}, \mathbf{h}') = \int_{\mathcal{S}} \mathbf{f} \mathbf{h}' d\Sigma$$

holds for every $\mathbf{h}' \in \mathbf{V}$. The equivalence between (3.4) and classical formulation is a fact of regularity.

4. Existence and Uniqueness Theorem. Lemma 4.1. *If the functions ${}_iC$, $i=0, 1, 2, 3, 4, 5$ are measurable and bounded on Ω , than the following inequality*

$$(4.1) \quad |A(\mathbf{h}, \mathbf{h}')| \leq \text{const.} \|\mathbf{h}\|_1 \|\mathbf{h}'\|_1$$

holds.

Proof. In view of (1.4) it is easily to see that

$$\max \{ |e_{\alpha\beta}|, |\kappa_{\alpha\beta}|, |\gamma_{\alpha}|, |s|, |s_{\alpha}| \} \leq \text{const.} \|\mathbf{h}\|_1.$$

As ${}_iC$ are bounded follows the inequality (4.1).

From Lemma 4.1 it results the continuity of the form $A(\mathbf{h}, \mathbf{h})$.

Let the operators $N_p \mathbf{h}$ be given by

$$\begin{aligned}
(4.2) \quad & N_1 \mathbf{h} = u_{1,1} - B_{11} u_3, \quad N_2 \mathbf{h} = \frac{1}{2} u_{1/2} + \frac{1}{2} u_{2/1} - B_{12} u_3, \quad N_3 \mathbf{h} = u_{2/2} - B_{22} u_3, \\
& N_4 \mathbf{h} = D_{,1} B_1^\gamma u_\gamma + D B_1^\beta B_{1\beta} u_3 - B_{11} \delta_3 - D B_1^\beta u_{\beta/1} + D_{,1} u_{3,1} + \delta_{1/1}, \\
& N_5 \mathbf{h} = D_{,2} B_2^\gamma u_\gamma + D B_2^\beta B_{2\beta} u_3 - B_{22} \delta_3 - D B_2^\beta u_{\beta/2} - D_{,2} u_{3,2} + \delta_{2/2}, \\
& N_6 \mathbf{h} = D_{,1} B_2^\gamma u_\gamma + D B_1^\beta B_{2\beta} u_3 - B_{12} \delta_3 - D B_1^\beta u_{\beta/2} + D_{,1} u_{3,2} + \delta_{2/1}, \\
& N_7 \mathbf{h} = D_{,2} B_1^\gamma u_\gamma + D B_2^\beta B_{1\beta} u_3 - B_{21} \delta_3 - D B_2^\beta u_{\beta/1} + D_{,2} u_{3,1} + \delta_{1/2}, \\
& N_8 \mathbf{h} = D^2 B_1^\gamma B_2^\delta u_\delta + D B_1^\gamma \delta_\gamma + D^2 B_1^\gamma u_{3,\gamma} + D \delta_{3,1} + D_{,1} \delta_3, \\
& N_9 \mathbf{h} = D^2 B_2^\gamma B_1^\delta u_\delta + D B_2^\gamma \delta_\gamma + D^2 B_2^\gamma u_{3,\gamma} + D \delta_{3,2} + D_{,2} \delta_3, \\
& N_{10} \mathbf{h} = D B_1^\gamma u_\gamma + \delta_1 + D u_{3,1}, \quad N_{11} \mathbf{h} = D B_2^\gamma u_\gamma + \delta_2 + D u_{3,2}, \quad N_{12} \mathbf{h} = 2 D \delta_3.
\end{aligned}$$

This operator can be written in the form ([4], Chap. 3, Eq. (7.12))

$$(4.3) \quad N_p \mathbf{h} = \sum_{s=1}^6 \sum_{|\alpha| \leq 1} a_{ps\alpha} D^\alpha h_s.$$

Definition 4.1. The form $b(\mathbf{h}, \mathbf{g})$ given by

$$(4.4) \quad b(\mathbf{h}, \mathbf{g}) = \int_{\mathcal{D}} \left(\sum_{i=1}^{12} N_i \mathbf{h} \cdot N_i \mathbf{g} \right) d\mathbf{x}$$

is called **V-coercive** if there exists a positive constant λ such that for every $\mathbf{h} \in \mathbf{V}$ we have

$$(4.5) \quad b(\mathbf{h}, \mathbf{h}) + \lambda \|\mathbf{h}\|_0^2 \geq c \|\mathbf{h}\|_1^2, \quad c = \text{constant}, \quad c > 0.$$

Definition 4.2. ([4] Chap. 3). Let $\xi \in R_2$ be an arbitrar vector and let N_{ps} be the matrix

$$(4.6) \quad N_{ps} \xi = \sum_{|\alpha|=1} a_{ps\alpha} (\theta^1, \theta^2) \xi^\alpha.$$

The operators (4.2) form, in a point, a complet algebraic system, if for $\xi \neq 0$ we have

$$(4.7) \quad \text{rank}(N_{ps} \xi) = 6.$$

Lemma 4.2. If the functions D and $\bar{B}_{\alpha\beta}$ are continucus, than the form (4.4) is **V-coercive**.

Proof. First we quote the following theorem ([4], Theorem, 7.7, p. 192): "Let Ω be with Lipschitz boundary and let the operators

$$N_p \mathbf{h} = \sum_{s=1}^6 \sum_{|\alpha| \leq 1} a_{ps\alpha} D^\alpha h_s,$$

with $a_{ps\alpha} \in C^0(\bar{\Omega})$ for $|\alpha|=1$. If for every point in $\bar{\Omega}$, the operators $N_p(\theta^1, \theta^2) \mathbf{h}$ form a complet algebraic system, than the form $b(\mathbf{h}, \mathbf{h})$ is **V-coercive**". As we have assumed that $\partial\Omega$ is Lipschitz, we must proof that the system (4.2) is algebraic complet. Writting the matrix N_{ps} for the operators we get

$$(4.8) \quad N_{ns\xi} = \begin{bmatrix} \xi^1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\xi^2 & \frac{1}{2}\xi^1 & 0 & 0 & 0 & 0 \\ 0 & \xi^2 & 0 & 0 & 0 & 0 \\ -DB_1^1\xi^1 & -DB_1^2\xi^1 & D_{,1}\xi^1 & \xi^1 & 0 & 0 \\ -DB_2^1\xi^2 & -DB_2^2\xi^2 & D_{,2}\xi^2 & 0 & \xi^2 & 0 \\ -DB_1^1\xi^2 & -DB_1^2\xi^2 & D_{,1}\xi^2 & 0 & \xi^1 & 0 \\ -DB_2^1\xi^2 & -DB_2^2\xi^2 & D_{,2}\xi^1 & \xi^2 & 0 & 0 \\ 0 & 0 & D^2B_1^1\xi^1 + D^2B_2^2\xi^2 & 0 & 0 & D\xi^1 \\ 0 & 0 & D^2B_2^1\xi^1 + D^2B_2^2\xi^2 & 0 & 0 & D\xi^2 \\ 0 & 0 & D\xi^1 & 0 & 0 & 0 \\ 0 & 0 & D\xi^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

If $\xi^1 \neq 0$ and $\xi^2 \neq 0$ than $\Delta_1 = (D)^2(\xi^1)^3(\xi^2)^3 \neq 0$. When $\xi^1 = 0$ and $\xi^2 \neq 0$ than $\Delta_2 = 2^{-1}(D)^2(\xi^2)^6 \neq 0$. Finally, when $\xi^1 \neq 0$ and $\xi^2 = 0$ than $\Delta_3 = 2^{-1}(D)^2(\xi^1)^6 \neq 0$. Since $D \neq 0$ ($D=0$ for the membrane theory), obviously that the matrix (4.8) has always the rank 6.

We suppose the energy deformation function to be positive definit, i.e. we have

$$(4.9) \quad A(\mathbf{h}, \mathbf{h}) \geq c \sum_{p=1}^{12} \|N_p \mathbf{h}\|_0^2, \quad c = \text{const.}, \quad c \geq 0.$$

Let P be the set

$$P = \left\{ \mathbf{h} \in \mathbf{V}, \sum_{p=1}^{12} \|N_p \mathbf{h}\|_0^2 = 0 \right\}$$

and $\mathbf{V}/P$ the space of the classes $\tilde{\mathbf{h}} = \mathbf{h} + \mathbf{p}$, $\mathbf{h} \in \mathbf{V}$, $\mathbf{p} \in P$ with the norm

$$\|\tilde{\mathbf{h}}\|_{\mathbf{V}/P} = \inf_{\mathbf{p} \in P} \|\mathbf{h} + \mathbf{p}\|_1.$$

Again we recall an useful theorem ([6], Theorem 2.1): „Let the bilinear form $A(\mathbf{h}, \mathbf{g})$ be definit on $\mathbf{V}/P \times \mathbf{V}/P$ and let the conditions (4.5) and (4.9) be satisfied. Then the following inequality

$$(4.10) \quad A(\tilde{\mathbf{h}}, \tilde{\mathbf{g}}) \geq c_1 \sum_{p=1}^{12} \|N_p \mathbf{h}\|_0^2 \geq c_2 \|\tilde{\mathbf{h}}\|_{\mathbf{V}/P}^2,$$

holds too, for every $\mathbf{h} \in \mathbf{V}/P$. Such a way, if $P \equiv 0$, obviously the sesquilinear form $A(\mathbf{h}, \mathbf{g})$ is $\mathbf{V}$ -elliptic ($|A(\mathbf{h}, \mathbf{h})| \geq \alpha \|\mathbf{h}\|_1^2$, $\alpha > 0$).

It is known [5] that if the measures $e_{\alpha\beta}$, $\varkappa_{\alpha\beta}$, γ_α , s_α , s are given, then we obtain the displacement $\mathbf{u}$, δ (if some compatibility conditions are fulfilled ([5], Eqs. (6.60), (6.61)) except a rigid motion given by

$$(4.11) \quad \mathbf{u} = \mathbf{c} + \omega_0 \times \mathbf{R}, \quad \delta = \omega_0 \times \mathbf{D},$$

where ω_0 is the following vector ([5], Eqs. (6.49), (6.29))

$$(4.12) \quad \omega_0^\alpha = \varepsilon^{\alpha\gamma} (u_{3,\alpha}(P_0) + B_4^\gamma u_\gamma(P_0)), \quad \omega_0^3 = -\frac{1}{4} \varepsilon^{\lambda\gamma} (u_{\lambda/2}(P_0) - u_{\gamma/2}(P_0)),$$

P_0 is an arbitrary fixed point, and ${}_0\varepsilon^{\nu\alpha}$ is given by

$$(4.13) \quad {}_0\varepsilon^{\alpha\beta} = A^{-1/2} e^{\alpha\beta}, \quad e^{11} = e^{22} = 0, \quad e^{12} = -e^{21} = 1.$$

Lemma 4.3. *If the displacement vector $\mathbf{h}$ satisfies the following boundary conditions*

$$(4.14) \quad \mathbf{h} \in \mathbf{V} = \mathcal{H}_0^1(\mathcal{G}) \quad \text{and} \quad u_{2/1} - u_{1/2} = 0 \quad \text{on} \quad \partial\Omega,$$

then it results that $P \equiv 0$.

Proof Since $\sum_{\alpha, \beta=1}^2 (e_{\alpha\beta}^2 + x_{\alpha\beta}^2) + \sum_{\alpha=1}^2 (\gamma_\alpha^2 + s_\alpha) + s^2 = 0$, it results that $\mathbf{u} = \mathbf{c} + \omega_0 \times \mathbf{R}$ and $\delta = \omega_0 \times \mathbf{D}$. Conforming with (4.14)₂ we obtain $\omega_0^3 = 0$. By using (4.14)₁ it results $\omega_0^1 = \omega_0^2 = c_1 = c_2 = c_3 = 0$ and hence $\mathbf{h} = 0$ in $\mathcal{G}$.

From all the preceding results we have the following existence and uniqueness.

Theorem (4.1). *If the following conditions are satisfied :*

- 1) *the energy deformation function is positive definit;*
- 2) *the coefficients c_i , $i=0, 1, 2, 3, 4, 5$ are measurable and bounded functions;*

3) *the thickness of the shell has a continuous variation ($D = D(\theta^1, \theta^2) \in C^1(\Omega)$);*

4) *the surface of the generalized shell has a continuous curvature ($B \in C^1(\Omega)$);*

5) *the boundary is fixed ($\mathbf{h} \in \mathbf{V} = \mathcal{H}_0^1(\mathcal{G})$) and is fulfilled the condition $u_{1/2} = u_{2/1}$,*

then the formulated boundary value problem has a unique weak solution.

Proof. The proof results in view of the Lax-Milgram Lemma [4] and the V -ellipticity conditions.

Received October 5, 1983

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SOLUȚII SLABE ÎN TEORIA GENERALIZATĂ A PLĂCILOR CURBE ANIZOTROPE CU GROSIME NEUNIFORMĂ

(Rezumat)

Se studiază problema la limită cu deplasări nule pe frontieră din teoria liniară generalizată a plăcilor curbe anizotrope cu grosime neuniformă. Presupunind că energia de deformare este pozitiv definită, coeficienții constitutivi sînt funcții măsurabile și mărginite, grosimea plăcii variază continuu, iar suprafața modelului generalizat de placă curbă are curbura continuă, se demonstrează existența și unicitatea soluției slabe.

SYSTEMATIC METHODS OF APPROXIMATION IN NONLINEAR
THEORY OF COSSERAT MEDIA (II)

BY

I. NISTOR

1. Introduction. In 1930 Signorini [1] sketched a formal perturbation method for solving a class of traction boundary-value problems of finite elasticity. Signorini has shown that when the total applied load is equipollent to zero but does not possess an axis of equilibrium, then a series expansion of elastic equations is unique, if it exists. When the applied tractions and body forces both contain a multiplying parameter ε , Stoppelli [2], [3] has given a proof of existence and uniqueness of solution of the general elastic equations and has shown that the displacement can be expanded as an absolutely convergent power series in ε with non-zero radius of convergence, provided ε is sufficiently small. Green and Spratt [4] assumed that the prescribed surface tractions and body forces depend analytically on the parameter ε and that they vanish when $\varepsilon=0$. Capriz and Guidugli [5] have given a perturbation scheme of Signorini's type for the initial value problem of dead traction in finite elastodynamics.

Nistor [6] has proved an existence and uniqueness theorem of Stoppelli's type in nonlinear theory of Cosserat media. Method of successive approximation is extended by Neagu [7] to nonlinear theory of simple dipolar elastic solids.

In this paper we will give a perturbation method of Signorini's type for nonlinear equations of Cosserat bodies. If the loads do not possess an axis of equilibrium, then the microrotation is unique while the displacement is unique within a uniform translation.

2. Basic Equations. Throughout this paper a rectangular coordinate system is employed. The deformation of a Cosserat medium is described by the functions [8]

$$(2.1) \quad y_i = y_i(x_k), \quad R_{ij} = R_{ij}(x_p),$$

where x_i and y_j are the coordinates of the place occupied by the body point in the reference configuration B_r and the equilibrium configuration B , respectively; R_{ij} are the components of an orthogonal tensor which measures the microrotation of the body point. The components of a proper orthogonal tensor may be written in the form [6]

$$(2.2) \quad R_{ij} = \frac{1}{1 + \frac{1}{4}\varphi_h\varphi_h} [(1 - \frac{1}{4}\varphi_k\varphi_k)\delta_{ij} + \frac{1}{2}\varphi_i\varphi_j + e_{jip}\varphi_p],$$

where φ_i are the components of the microrotation vector, e_{ijk} is the alternating symbol.

We consider as strain tensors

$$(2.3) \quad e_{ij} = y_{k,i}R_{kj} - \delta_{ij}, \quad \gamma_{ij} = \frac{1}{2}e_{jmn}R_{pn}R_{pm,i},$$

where the symbol “ $_{,i}$ ” denotes partial derivation with respect to the variable x_i .

Using (2.2), the deformation tensor γ_{ij} may be represented in the form

$$(2.4) \quad \gamma_{ij} = H_{jp}\varphi_{p,i},$$

where

$$(2.5) \quad H_{jp} = \frac{1}{2(1 + \frac{1}{4}\varphi_k\varphi_k)} (2\delta_{jp} + e_{jpq}\varphi_q).$$

The equations of equilibrium of Cosserat media are [8]

$$(2.6) \quad (R_{ij}t_{kj})_{,k} + \rho F_i = 0, \\ (R_{ij}m_{kj})_{,k} + e_{ijk}y_{j,p}R_{kp}t_{pq} + \rho e_{ijk}R_{jp}B_{pk} = 0,$$

where

$$(2.7) \quad t_{ij} = \frac{\partial W}{\partial e_{ij}}, \quad m_{kj} = \frac{\partial W}{\partial \gamma_{kj}}, \quad W = W(e_{ij}, \gamma_{pq}).$$

In these equations we have used the following notations: t_{ij} and m_{ij} — components of stress tensor and couple-stress tensor, respectively, measured per unit area in reference configuration; F_i — components of the body force vector; B_{jk} — components of the body generalized forces vectors; ρ — density of the body in reference configuration; W — strain energy function per unit volume in reference configuration.

In the case of prescribed values of the surface forces f_i and generalized surfaces forces b_{ij} , the boundary conditions are written as

$$(2.8) \quad R_{ij}t_{kj}N_k = f_i, \quad R_{ij}m_{kj}N_k = e_{ijk}R_{jp}b_{pk},$$

where N_i are the components of the outward unit normal to the surface ∂B_r .

Any solution of boundary value-problem (2.6), (2.8) must be such that the external loads in the actual configuration must be equilibrated. These conditions are equivalent to [8]

$$(2.9) \quad \int_{B_r} \rho F_i dv + \int_{\partial B_r} f_i ds = 0,$$

$$(2.10) \quad e_{ijk} \left[\int_{B_r} \rho (y_j F_k + R_{jp} B_{pk}) dv + \int_{\partial B_r} (f_k y_j + R_{jp} b_{pk}) ds \right] = 0,$$

where y_j are the components of the place occupied by the body point in the configuration of equilibrium but now are expressed as functions of x_i .

As we remarked in [8], without loss of generality we may assume that external loads satisfy the conditions

$$(2.11) \quad e_{ijk} \left[\int_{B_r} \rho(x_j F_k + B_{jk}) dv + \int_{\partial B_r} (x_j f_k + b_{jk}) ds \right] = 0.$$

Subtracting (2.11) from (2.10) we obtain that the conditions of compatibility may be written in the form

$$(2.12) \quad e_{ijk} \left\{ \int_{B_r} \rho[u_j F_k + (R_{jp} - \delta_{jp}) B_{pk}] dv + \int_{\partial B_r} [u_j f_k + (R_{jp} - \delta_{jp}) b_{pk}] ds \right\} = 0,$$

where $u_i = y_i - x_i$, are the components of the displacement vector.

3. Theorem of Uniqueness and Compatibility of Signorini's Type.

In the systematic method of approximation to be developed, we assume that the body forces F_i , generalized body forces B_{ij} , the surface forces f_i and generalized surface forces b_{kj} depend analitically on a parameter ε and that they vanish when $\varepsilon=0$. Thus we set up a one-parameter family of boundary-value problems corresponding to one parameter family of external loads given by the series expansions

$$(3.1) \quad F_i = \sum_{n=1}^{\infty} \varepsilon^n F_i^{(n)}, \quad f_i = \sum_{n=1}^{\infty} \varepsilon^n f_i^{(n)}, \quad B_{ij} = \sum_{n=1}^{\infty} \varepsilon^n B_{ij}^{(n)}, \quad b_{ij} = \sum_{n=1}^{\infty} \varepsilon^n b_{ij}^{(n)}.$$

Corresponding to this one-parameter family of boundary-value problems we seek a one-parameter family of solutions (u_i, φ_i) which depend analitically on the parameter ε and vanish when $\varepsilon=0$

$$(3.2) \quad u_i = \sum_{n=1}^{\infty} \varepsilon^n u_i^{(n)}, \quad \varphi_i = \sum_{n=1}^{\infty} \varepsilon^n \varphi_i^{(n)}.$$

In the specification of a particular problem the parameter may arise naturally as a quantity which may be increased from zero such as twist or some other strain.

Substitution of (3.2) into (2.2), (2.3)₁ and (2.4) shows that the micro-rotation tensor R_{ij} and the strain tensors e_{ij} , γ_{kj} have series expansions

$$(3.3) \quad R_{ij} = \delta_{ij} + \sum_{n=1}^{\infty} \varepsilon^n R_{ij}^{(n)}, \quad e_{ij} = \sum_{n=1}^{\infty} \varepsilon^n e_{ij}^{(n)}, \quad \gamma_{ij} = \sum_{n=1}^{\infty} \varepsilon^n \gamma_{ij}^{(n)},$$

where

$$(3.4) \quad R_{ij}^{(n)} = e_{jip} \varphi_p^{(n)} + \frac{n - \delta_{2n}}{2n} (\varphi_i^{(1)} \varphi_j^{(n-1)} + \varphi_i^{(n-1)} \varphi_j^{(1)} - 2\delta_{ij} \varphi_q^{(1)} \varphi_q^{(n-1)}) + \dots,$$

$$\gamma_{ij}^{(n)} = \varphi_{j,i}^{(n)} + \frac{n - \delta_{2n}}{2n} e_{jipq} (\varphi_{p,i}^{(n-1)} \varphi_q^{(1)} + \varphi_{p,i}^{(1)} \varphi_q^{(n-1)}) + \dots,$$

$$e_{ij}^{(n)} = u_{j,i}^{(n)} + e_{jip} \varphi_p^{(n)} + \frac{n - \delta_{2n}}{n} e_{jkp} (u_{k,i}^{(n-1)} \varphi_p^{(1)} + u_{k,i}^{(1)} \varphi_p^{(n-1)}) + \\ + \frac{n - \delta_{2n}}{2n} (\varphi_i^{(1)} \varphi_j^{(n-1)} + \varphi_j^{(1)} \varphi_i^{(n-1)} - 2\delta_{ij} \varphi_p^{(1)} \varphi_p^{(n-1)}) + \dots$$

The terms neglected in the relations (3.4) do not depend upon $u_i^{(n)}$, $u_i^{(n-1)}$, $\varphi_i^{(n)}$, $\varphi_i^{(n-1)}$.

We assume that the reference configuration is a natural state for every point of the Cosserat body and that the strain energy W is an analytic function of e_{ij} , γ_{kj} , so that

$$(3.5) \quad W = \sum_{n=1}^{\infty} \varepsilon^n W^{(n)}.$$

Substitution (3.5) into (2.7) yields the corresponding series expansions of the stress tensor t_{ij} and couple-stress tensor m_{ij}

$$(3.6) \quad t_{ij} = \sum_{n=1}^{\infty} \varepsilon^n t_{ij}^{(n)}, \quad m_{ij} = \sum_{n=1}^{\infty} \varepsilon^n m_{ij}^{(n)},$$

where

$$(3.7) \quad t_{ij}^{(n)} = \frac{1}{n!} \frac{d^n}{d\varepsilon^n} \frac{\partial W}{\partial e_{ij}}, \quad m_{ij}^{(n)} = \frac{1}{n!} \frac{d^n}{d\varepsilon^n} \frac{\partial W}{\partial \gamma_{ij}}.$$

It is easy to verify that

$$(3.8) \quad \frac{d^n}{d\varepsilon^n} \frac{\partial W}{\partial e_{ij}} = \frac{1}{n+1} \frac{\partial}{\partial e_{ij}} \frac{d^{n+1} W}{d\varepsilon^{n+1}}, \quad \frac{d^n}{d\varepsilon^n} \frac{\partial W}{\partial \gamma_{ij}} = \frac{1}{n+1} \frac{\partial}{\partial \gamma_{ij}} \frac{d^{n+1} W}{d\varepsilon^{n+1}}.$$

Taking into account the relation (3.8), the relations (3.7) may be written in the form

$$(3.9) \quad t_{ij}^{(n)} = \tilde{t}_{ij}^{(n)} + \hat{t}_{ij}^{(n)}, \quad m_{ij}^{(n)} = \tilde{m}_{ij}^{(n)} + \hat{m}_{ij}^{(n)},$$

where

$$(3.10) \quad \tilde{t}_{ij}^{(n)} = \frac{\partial^2 W}{\partial e_{ij} \partial e_{pq}} (u_{q,p}^{(n)} + e_{qp} \varphi_p^{(n)}) + \frac{\partial^2 W}{\partial e_{ij} \partial \gamma_{pq}} \varphi_{q,p}^{(n)}, \\ \tilde{m}_{ij}^{(n)} = \frac{\partial^2 W}{\partial \gamma_{ij} \partial e_{pq}} (u_{q,p}^{(n)} + e_{qp} \varphi_p^{(n)}) + \frac{\partial^2 W}{\partial \gamma_{ij} \partial \gamma_{pq}} \varphi_{q,p}^{(n)},$$

and the terms $\hat{t}_{ij}^{(n)}$, $\hat{m}_{ij}^{(n)}$ depend on $u_i^{(1)}$, $u_i^{(2)}$, ..., $u_i^{(n-1)}$, $\varphi_j^{(1)}$, ..., $\varphi_k^{(n-1)}$.

Substitution of the series (3.1), (3.3) and (3.6) into (2.6) and (2.8) and then equating like powers of ε yields the following successive system of differential equations and associated boundary conditions:

$$(3.11) \quad \tilde{t}_{ki,k}^{(n)} + \rho \tilde{F}_i^{(n)} = 0, \quad m_{ki,k}^{(n)} + e_{ijk} \tilde{t}_{jk}^{(n)} + \rho \tilde{G}_i^{(n)} = 0, \\ \tilde{t}_{ki}^{(n)} N_k = \tilde{f}_i^{(n)}, \quad \tilde{m}_{ki}^{(n)} N_k = \tilde{g}_i^{(n)},$$

where

$$\begin{aligned}
 \rho \tilde{F}_i^{(n)} &= \rho F_i^{(n)} + \hat{t}_{ki}^{(n)} + \sum_{\alpha=1}^{n-1} [t_{ij}^{(\alpha)} R_{ij}^{(n-\alpha)}]_{,k}, \\
 \rho \tilde{G}_i^{(n)} &= \hat{m}_{ki}^{(n)} + e_{ijk} \left[\sum_{\alpha=0}^{n-1} \rho R_{jp}^{(\alpha)} B_{pk}^{(n-\alpha)} + \hat{t}_{jk}^{(n)} + \sum_{\alpha=1}^{n-1} t_{pq}^{(\alpha)} \sum_{\beta=0}^{n-\alpha} u_{j,p}^{(\beta)} R_{kq}^{(n-\alpha-\beta)} \right] + \\
 (3.12) \quad &+ \sum_{\alpha=1}^{n-1} [R_{ij}^{(\alpha)} m_{kj}^{(n-\alpha)}]_{,k}, \\
 \tilde{f}_i^{(n)} &= f_i^{(n)} - \hat{t}_{ki}^{(n)} N_k - \sum_{\alpha=1}^{n-1} [t_{kj}^{(\alpha)} R_{ij}^{(n-\alpha)}]_{,k}, \\
 \tilde{g}_i^{(n)} &= e_{ijk} \sum_{\alpha=0}^{n-1} R_{jq}^{(\alpha)} B_{qk}^{(n-\alpha)} - \hat{m}_{ki}^{(n)} N_k - \sum_{\alpha=1}^{n-1} R_{ij}^{(\alpha)} m_{kj}^{(n-\alpha)} N_k.
 \end{aligned}$$

When $n=1$, since $\tilde{F}_i^{(1)}=F_i^{(1)}$, $\tilde{G}_i^{(1)}=e_{ijk}B_{jk}^{(1)}$, $\tilde{f}_j^{(1)}=f_j^{(1)}$, $\tilde{g}_j^{(1)}=e_{ijk}b_{ik}^{(1)}$, the equations (3.11) are precisely those which define the boundary-value problem of traction and couples in the infinitesimal theory of micropolar elasticity, where $F_i^{(1)}$, $\tilde{G}_j^{(1)}$, $f_j^{(1)}$, $\tilde{g}_k^{(1)}$, are the assigned external body force, body couple, surface traction and surface couple, respectively. Let be supposed that fields $u_i^{(1)}$, $u_j^{(2)}$, ..., $u_j^{(p-1)}$, $\varphi^{(1)}$, $\varphi^{(2)}$, ..., $\varphi^{(p-1)}$ satisfying (3.11) for $n=1, 2, \dots, p-1$ have been determined. Then for $n=p$ (3.11) has the form of the equations defining the boundary-value problem of traction and couple of the infinitesimal theory of micropolar elasticity for the same material and the same boundary though subject to assigned external body force, body couple, surface traction and surface couple which depend in an explicitly known way upon the previously determined fields $u_i^{(1)}$, $u_j^{(2)}$, ..., $u_j^{(p-1)}$, $\varphi^{(1)}$, $\varphi^{(2)}$, ..., $\varphi^{(p-1)}$. Thus the calculation of p terms in the expansions (3.2) is reduced formally to the solution of p boundary value problems of traction and couple in the infinitesimal theory of micropolar elasticity for the same body.

If we substitute (3.1)–(3.3) into (2.12) and expand, we obtain

$$(3.13) \quad e_{ijk} \sum_{\alpha=1}^{n-1} \left\{ \int_{B_r} \rho [u_j^{(n-\alpha)} F_k^{(\alpha)} + R_{jq}^{(n-\alpha)} B_{qk}^{(\alpha)}] dv + \int_{\partial B_r} [u_j^{(n-\alpha)} f_k^{(\alpha)} + R_{jq}^{(n-\alpha)} b_{qk}^{(\alpha)}] ds \right\} = 0.$$

Thus, in order that the expansion (3.2) shall give a solution, it is necessary not only that $(u_i^{(n)}, \varphi_j^{(n)})$ be a solution of the system (3.11), but also that they satisfy the conditions of compatibility (3.13).

In order to determine $(u_i^{(n)}, \varphi_j^{(n)})$ inductively, so suppose that solutions $u_i^{(1)}$, $u_i^{(2)}$, ..., $u_i^{(p-1)}$, $\varphi^{(1)}$, $\varphi^{(2)}$, ..., $\varphi^{(p-1)}$ of (3.11) for $n=1, 2, \dots, p-1$ have already been found such that the compatibility conditions (3.13) are satisfied for $n=2, 3, \dots, p$. Let $(u_i^{(p)}, \varphi_j^{(p)})$ be any solution of (3.11) for $n=p$ and put

$$\begin{aligned}
 r_i^{(p)} &= e_{ijk} \sum_{h=1}^p \left\{ \int_{B_r} \rho [u_j^{(p+1-h)} F_k^{(h)} + R_{jq}^{(p+1-h)} B_{qk}^{(h)}] dv + \right. \\
 &+ \left. \int_{\partial B_r} [u_j^{(p+1-h)} f_k^{(h)} + R_{jq}^{(p+1-h)} b_{qk}^{(h)}] ds \right\} = 0.
 \end{aligned}$$

If $c_i^{(p)}$ are the components of any constant vector, then $(u_i^{(p)} + e_{ijk}x_j c_k^{(p)}, \varphi_i^{(p)} + c_i^{(p)})$ is also a solution of (3.11) for $n=p$ and this solution will satisfy the compatibility condition (3.13) if

$$(3.14) \quad (A_{jk}^{(1)} - \delta_{jk} A_{ii}^{(1)}) c_j^{(p)} = r_k^{(p)},$$

where $A_{jk}^{(1)}$ are the components of the astatic load tensor [8] corresponding to $(F_i^{(1)}, B_{jk}^{(1)}, f_k^{(1)}, b_{pq}^{(1)})$.

In order that (3.14) posses an unique solution it is necessary and sufficient that

$$(3.15) \quad \det(A_{ij} - \delta_{ij} A_{hh}) \neq 0.$$

As is known [8], that the load $(F_i^{(1)}, B_{jk}^{(1)}, f_i^{(1)}, b_{pq}^{(1)})$ must not posses an axis of equilibrium. Under this condition, therefore, $(u_i^{(n)}, \varphi_j^{(n)})$ are unique.

As remarked at the begining of this section, the system (3.11) since it has the form of the equations governing the traction and couple boundary-value problem of infinitesimal micropolar elasticity determines $(u_i^{(n)}, \varphi_j^{(n)})$ at most to within an uniform infinitesimal rotation. If (3.15) holds, then this rotation is uniquely determined by the compatibility condition (3.13).

In order that the system (3.11) have a solution $(u_i^{(n)}, \varphi_j^{(n)})$ at all it is necessary that the effective loads $\tilde{F}_i^{(n)}, \tilde{G}_i^{(n)}, \tilde{f}_j^{(n)}, \tilde{g}_j^{(n)}$ be equilibrated, i.e. that (2.9) and (2.11) hold if $(F_i, e_{ijk}B_{jk}, f_j, e_{ijk}b_{jk})$ are replaced by $(\tilde{F}_j^{(n)}, \tilde{G}_i^{(n)}, \tilde{f}_k^{(n)}, \tilde{g}_j^{(n)})$. Now, if we substitute the expansion (3.1) into (2.9) and (2.11) we obtain

$$(3.16) \quad \int_{B_r} \rho F_i^{(n)} dv + \int_{\partial B_r} f_i^{(n)} ds = 0, \\ e_{ijk} \left[\int_{B_r} \rho (x_j F_k^{(n)} + B_{jk}^{(n)}) dv + \int_{\partial B_r} (x_j f_k^{(n)} + b_{jk}^{(n)}) ds \right] = 0.$$

Taking into account the relation (3.16), the conditions for the existence of a solution $(u_i^{(n)}, \varphi_j^{(n)})$ of (3.11) are equivalent to

$$(3.17) \quad \int_{B_r} \rho (\tilde{F}_i^{(n)} - F_i^{(n)}) dv + \int_{\partial B_r} (\tilde{f}_i^{(n)} - f_i^{(n)}) ds = 0,$$

$$(3.18) \quad e_{ijk} \left\{ \int_{B_r} \rho [x_j (\tilde{F}_k^{(n)} - F_k^{(n)}) + \tilde{G}_i^{(n)} - G_i^{(n)}] dv + \right. \\ \left. + \int_{\partial B_r} [x_j (\tilde{f}_k^{(n)} - f_k^{(n)}) + \tilde{g}_i^{(n)} - g_i^{(n)}] ds \right\} = 0,$$

where

$$G_i^{(n)} = e_{ijk} B_{jk}^{(n)}, \quad g_i^{(n)} = e_{ijk} b_{jk}^{(n)}.$$

Use of (3.12) and the divergence theorem shows easily that (3.17) is satisfied and (3.18) is equivalent to

$$(3.19) \quad e_{ijk} \sum_{\alpha=1}^{n-1} \left\{ \int_{B_r} \rho \left[R_{jq}^{(\alpha)} B_{qk}^{(n-\alpha)} + u_{j,p}^{(n-\alpha)} \sum_{\beta=0}^{\alpha-1} R_{kq}^{(\beta)} t_{pq}^{(n-\beta)} \right] dv + \int_{\partial B_r} R_{jq}^{(\alpha)} b_{qk}^{(n-\alpha)} ds \right\} = 0.$$

Using the equation (3.11) for $n=\alpha$, the relation (3.19) becomes

$$(3.20) \quad e_{ijk} \sum_{\alpha=1}^{n-1} \left\{ \int_{B_r} \rho [u_j^{(n-\alpha)} F_k^{(\alpha)} + R_{jp}^{(n-\alpha)} B_{pk}^{(\alpha)}] dv + \int_{\partial B_r} [u_j^{(n-\alpha)} f_k^{(\alpha)} + R_{jq}^{(n-\alpha)} b_{qk}^{(\alpha)}] ds \right\} = 0.$$

We see that (3.20) is the compatibility condition (3.13). Thus these compatibility conditions are just the right restrictions to be imposed in order that (3.11) can have any solution at all.

The result established in this section may then be stated as the following theorem of compatibility and uniqueness.

Theorem. *Let it be supposed that the loads $(F_i^{(1)}, B_{jk}^{(1)}, f_h^{(1)}, b_{pq}^{(1)})$, do not possess an axis of equilibrium. Then*

1. *The system (3.11), which must be satisfied if (3.2) is to be a solution, is compatible; i.e. if solutions exist for the traction and couple boundary-value problem in the infinitesimal theory of micropolar elasticity, then there exists solutions $(u_i^{(1)}, \varphi_i^{(1)}), (u_j^{(2)}, \varphi_j^{(2)}), \dots$, of the system (3.11).*

2. *If the classical uniqueness theorem for the traction and couple boundary-value problem in the linear theory holds, then $\varphi^{(1)}, \varphi^{(2)}, \dots$ are unique, while $u_i^{(1)}, u_j^{(2)}, \dots$ are unique to within a uniform translation.*

If the loads $(F_i^{(1)}, B_{ij}^{(1)}, f_k^{(1)}, h_{pq}^{(1)})$ have an axis of equilibrium then (3.14) does not determine the unique infinitesimal rotation $c_i^{(p)}$; system (3.14) may now have no solution or more than one.

In classical elasticity of finite deformation Tolotti [9] has given an exhaustive analysis of the various possible cases when there is an axis of equilibrium and the successive system are compatible for every n , for n up to a certain value, or for no n .

Received June 18, 1981

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METODE SISTEMATICE DE APROXIMARE ÎN TEORIA NELINIARĂ A MEDIILOR COSSERAT (II)

(Rezumat)

Se dă o metodă de perturbare de tip Signorini pentru mediile Cosserat. Dacă încărcările exterioare nu au axă de echilibru, atunci microrotația este unic determinată iar deplasarea este determinată până la o translație uniformă.

ON THE UNSTEADY FLOW OF VISCO-ELASTIC MAXWELL FLUID
THROUGH A TUBE OF ELLIPTIC SECTION

BY

S. LAHIRI and S. K. DHAR

1. Introduction. The flow of Newtonian viscous fluid through a long tube of elliptic section under a periodic pressure gradient has been studied by Khamrui [1]. In recent years the unsteady flow of elastico-viscous liquids through tubes and channels has attracted the attention of many investigators. In the present paper the flow of a visco-elastic Maxwell fluid through a long tube of elliptic section under the influence of a periodic pressure gradient has been considered. The solution is obtained in terms of Mathieu functions and from it the results in the two extreme cases of very small and very large frequencies are deduced.

For very small frequencies, the velocity has the same phase as the pressure gradient and the motion at each instant is the same as if the pressure gradient is constant. For very large frequencies the motion has the boundary layer character.

The rheological equations of a visco-elastic fluid of the Maxwell type (that is, a spring and a dash-pot arranged in series) are characterized by the equations

$$(1) \quad \begin{aligned} \tau_{ij} &= -p\delta_{ij} + \tau'_{ij}, \\ \left(1 + \lambda \frac{\partial}{\partial t}\right) \tau'_{ij} &= 2\mu e_{ij}, \\ e_{ij} &= \frac{1}{2}(v_{i,j} + v_{j,i}), \end{aligned}$$

where τ'_{ij} is the deviatoric stress tensor; e_{ij} — the rate-of-strain tensor; p — the pressure; λ, μ are material constants (λ being of the dimension of time, is called the relaxation time and μ is the viscous parameter) and v_i ($i=1, 2, 3$) are the velocity components. The behaviour of the fluid is more like an elastic solid when λ is large, and when λ is small, the behaviour is like a viscous fluid.

2. Fundamental Equations and Solution. Let us take the z -axis along the axis of the tube. If the tube is long enough, the velocity ω along it is independent of z and the other components of velocity are assumed to be zero. With the help of (1), the equations of motion become

$$(2) \quad 0 = -\frac{1}{\rho} \left(1 + \lambda \frac{\partial}{\partial t} \right) \frac{\partial p}{\partial x},$$

$$(3) \quad 0 = -\frac{1}{\rho} \left(1 + \lambda \frac{\partial}{\partial t} \right) \frac{\partial p}{\partial y},$$

and

$$(4) \quad \left(1 + \lambda \frac{\partial}{\partial t} \right) \frac{\partial \omega}{\partial t} = -\frac{1}{\rho} \left(1 + \lambda \frac{\partial}{\partial t} \right) \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right).$$

From (2) and (3) it is clear that p is a function of z and t only.

If the pressure gradient is periodic with periodic $2\pi/\sigma$, let us assume

$$-\frac{1}{\rho} \frac{\partial p}{\partial z} = \alpha \exp(i\sigma t),$$

and

$$\omega = \omega_1(x, y) \exp(i\sigma t),$$

where α is a real constant and the real parts of the right hand sides are taken.

Substituting in (4) and putting

$$v = \omega_1 + \frac{i\alpha}{\sigma}, \quad h^2 = \frac{i\sigma}{\nu} (1 + i\lambda\sigma),$$

we get

$$(5) \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - h^2 v = 0.$$

If the boundary of the tube is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we introduce elliptic co-ordinates ξ, η defined by

$$x + iy = c \cosh(\xi + i\eta),$$

where

$$c = (a^2 - b^2)^{\frac{1}{2}}.$$

It can be easily shown that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{2}{c^2 (\cosh 2\xi - \cos 2\eta)} \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right),$$

so that Eq. (5) transforms to

$$(6) \quad \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} - 2k^2 (\cosh 2\xi - \cos 2\eta) v = 0,$$

where

$$4k^2 = h^2 c^2 = \frac{i\sigma c^2}{\nu} (1 + i\lambda\sigma).$$

Substituting $v = \Phi(\xi)\Psi(\eta)$ in (6) and separating the variables in the usual manner, we are led to modified Mathieu equation for $\Phi(\xi)$ and the Mathieu equation for $\Psi(\eta)$, namely

$$(7) \quad \frac{d^2\Phi}{d\xi^2} - (a + 2k^2 \cosh 2\xi)\Phi = 0,$$

$$(8) \quad \frac{d^2\Psi}{d\eta^2} + (a + 2k^2 \cos 2\eta)\Psi = 0,$$

where a is the separation constant.

Here we shall consider the case when the flow distribution is symmetrical about the axis of the ellipse, so that $\Psi(\eta)$ must be even and periodic in η with period π . Hence the appropriate general solution of (6) is

$$(9) \quad v = \sum_{n=0}^{\infty} C_{2n} \text{ce}_{2n}(\xi, -k^2) \text{ce}_{2n}(\eta, -k^2),$$

where $\text{ce}_{2n}(\eta, -k^2)$ is the Mathieu function of the first kind of order $2n$ and $\text{Ce}_{2n}(\xi, -k^2)$ is the modified Mathieu function of the first kind and of order $2n$.

Therefore

$$(10) \quad \omega_1 = -\frac{i\alpha}{\sigma} + \sum_{n=0}^{\infty} C_{2n} \text{ce}_{2n}(\xi, -k^2) \text{ce}_{2n}(\eta, -k^2).$$

We have (McLachlan [2], pp. 21 and 27)

$$(11) \quad \text{ce}_{2n}(\eta, -k^2) = (-1)^n \text{ce}_{2n}\left(\frac{\pi}{2} - \eta, k^2\right) = (-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} \cos 2r\eta,$$

$$\text{Ce}_{2n}(\xi, -k^2) = \text{ce}_{2n}(i\xi, -k^2) = (-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} \cosh 2r\xi,$$

where the co-efficients $A_{2r}^{(2n)}$ are functions of k^2 .

If $\xi = \xi_0$ be the boundary of the elliptic section, we have the boundary condition

$$\omega_1 = 0, \quad \text{when} \quad \xi = \xi_0$$

i.e.

$$(12) \quad \frac{i\alpha}{\sigma} = \sum_{n=0}^{\infty} C_{2n} \text{ce}_{2n}(\xi_0, -k^2) \text{ce}_{2n}(\eta, -k^2).$$

Now multiplying both sides of (12) by $\text{ce}_{2n}(\eta, -k^2)$ and integrating with respect to η from 0 to 2π and using the first of the results in (11) and the orthogonality relations ([2], p. 321), we obtain

$$(13) \quad C_{2n} = \frac{2\pi (-1)^n A_0^{(2n)}}{\text{Ce}_{2n}(\xi_0, -k^2) L_{2n}} \frac{i\alpha}{\sigma},$$

where

$$(14) \quad L_{2n} = \int_0^{2\pi} ce_{2n}^2(\eta, -k^2) d\eta.$$

Therefore

$$(15) \quad \omega = \frac{\eta}{\sigma} \sin \sigma t + R \left[\exp(i \sigma t) \sum_{n=0}^{\infty} C_{2n} ce_{2n}(\xi, -q) ce_{2n}(\eta, -q) \right],$$

where R denotes the real part and $q = k^2$.

When the frequency of the pressure gradient is small, $|q|$ is small and we get ([2], p. 15)

$$ce_0(\eta, -q) \simeq 1 + \frac{1}{2} q \cos 2\eta,$$

$$ce_2(\eta, -q) \simeq \cos 2\eta + q \left(\frac{1}{12} \cos 4\eta - \frac{1}{4} \right),$$

with similar expressions for the modified functions $Ce_0(\xi, -q)$ and $Ce_2(\xi, -q)$.

Using the results (McLachlan [2], p. 46)

$$A_0^{(0)} = 1, \quad A_0^{(2)} = \frac{1}{4} q + O(q^3), \quad A_0^{(2n)} = O(q^n),$$

we have

$$C_0 = \frac{i\alpha}{\sigma} \left(1 - \frac{1}{2} q \cosh 2\xi_0 \right),$$

$$C_2 = -\frac{i\alpha}{2\sigma} \frac{q}{\cosh 2\xi_0}.$$

Hence substituting in (15), we obtain

$$(16) \quad \omega = \frac{\alpha c^2}{8\nu} \left[\cosh 2\xi_0 - \cosh 2\xi - \cos 2\eta + \frac{\cosh 2\xi \cos 2\eta}{\cosh 2\xi_0} \right] \times \\ \times (\cos \sigma t - \lambda \sigma \sin \sigma t).$$

In terms of Cartesian co-ordinates and using

$$a = c \cosh \xi_0, \quad b = c \sinh \xi_0,$$

we get

$$(17) \quad \omega = \frac{\alpha}{2\nu} \frac{a^2 b^2}{a^2 + b^2} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) (\cos \sigma t - \lambda \sigma \sin \sigma t).$$

Putting $\lambda = 0$ in (17) we see that this value of ω agrees with that obtained by K h a m r u i [1] for ordinary viscous liquid. So, we observe that when the frequency of the pressure gradient is very small, the motion at each instant is the same as the laminar motion in the tube under a constant pressure gradient, and this pressure gradient varies slowly with time.

In the case of a circular cylinder, $a = b$ and (17) reduces to

$$(18) \quad \omega = \frac{\alpha}{4\nu} (a^2 - r^2) (\cos \sigma t - \lambda \sigma \sin \sigma t),$$

where $r^2 = x^2 + y^2$.

When the frequency of the pressure gradient is very small, the total rate of flow through the elliptic tube is given by the integral of ω taken over the area of the cross-section. So

$$Q = \iint \omega dx dy = \frac{\alpha}{4\nu} \frac{\pi a^3 b^3}{a^2 + b^2} (\cos \sigma t - \lambda \sigma \sin \sigma t).$$

The maximum velocity occurs at the centre of the elliptic section and is given by

$$\omega_0 = \frac{\alpha}{2\nu} \frac{a^2 b^2}{a^2 + b^2} (\cos \sigma t - \lambda \sigma \sin \sigma t).$$

The average velocity over a cross-section is

$$\frac{Q}{\pi ab} = \frac{\alpha}{4\nu} \frac{a^2 b^2}{a^2 + b^2} (\cos \sigma t - \lambda \sigma \sin \sigma t).$$

Thus the average velocity is half the maxima velocity — a result which is in complete agreement with the corresponding result in the case of flow of ordinary viscous liquid when $\lambda = 0$.

For very large frequency of the pressure gradient, $|q|$ is very large and we get ([2], p. 47) $A_0^{(0)} = 1$ and $|A_0^{(2n)}|$, $n \geq 1$ are very small.

When $|q|$ is large, we have the asymptotic formula ([2], pp. 165, 230)

$$Ce_0(\xi, -q) \sim \left(\frac{2}{i \sinh \xi} \right)^{\frac{1}{2}} K_0 \cosh \left[2k \cosh \xi - \tanh^{-1} \tan \left(\frac{\pi}{4} - \frac{1}{2} i \xi \right) \right],$$

where

$$K_0 = \frac{ce_0(0) ce_0(\pi/2)}{A_0^{(0)} (2\pi k)^{1/2}}.$$

Since

$$4k^2 = \frac{i\sigma c^2}{\nu} (1 + i\sigma\lambda),$$

we get

$$2k = \beta k(\alpha + i\gamma)$$

with

$$\alpha^2 = \frac{1}{2} [-\lambda\sigma + (1 + \lambda^2\sigma^2)^{\frac{1}{2}}], \quad \gamma^2 = \frac{1}{2} [\lambda\sigma + (1 + \lambda^2\sigma^2)^{\frac{1}{2}}]$$

and

$$\beta = \left(\frac{\sigma}{\nu} \right)^{\frac{1}{2}}.$$

Now

$$\begin{aligned} & \cosh \left[2k \cosh \xi - \tanh^{-1} \left\{ \tan \left(\frac{\pi}{4} - \frac{1}{2} i \xi \right) \right\} \right] \simeq \\ & \simeq \frac{1}{2} \exp \left[\beta c(\alpha + i\gamma) \cosh \xi - \tanh^{-1} \left\{ \tan \left(\frac{\pi}{4} - \frac{1}{2} i \xi \right) \right\} \right] = \\ & = \frac{1}{2} (\tanh \frac{1}{2} \xi)^{\frac{1}{2}} \exp \{ \beta c(\alpha + i\gamma) \cosh \xi \}. \end{aligned}$$

Therefore for large values of $|q|$, we have from (10)

$$\omega_1 = -\frac{i\alpha}{\sigma} + \frac{i\alpha}{\sigma} \frac{Ce_0(\xi, -q)}{Ce_0(\xi_0, -q)} ce_0(\eta, -q) \sim \\ \sim -\frac{i\alpha}{\sigma} + \frac{i\alpha}{\sigma} \frac{\cosh \frac{1}{2}\xi_0}{\cosh \frac{1}{2}\xi} \exp \{-\beta c(\alpha + i\gamma)(\cosh \xi_0 - \cosh \xi)\} ce_0(\eta, -q).$$

Hence

$$(19) \quad \omega = \frac{\alpha}{\sigma} \sin \sigma t + \frac{\alpha}{\sigma} \frac{\cosh \frac{1}{2}\xi_0}{\cosh \frac{1}{2}\xi} \exp \{-\alpha\beta c(\cosh \xi_0 - \cosh \xi)\} \times \\ \times R[i \exp i\{\sigma t - \beta\gamma c(\cosh \xi_0 - \cosh \xi)\} ce_0(\eta, -q)],$$

where R denotes the real part.

For flow through a circular cylinder of radius a (19) reduces to

$$(20) \quad \omega = \frac{\alpha}{\sigma} \sin \sigma t - \frac{\alpha}{\sigma} \left(\frac{a}{r}\right)^{\frac{1}{2}} \exp \{-\alpha\beta(a-r)\} \sin \{\sigma t - \alpha\beta(a-r)\}.$$

3. Discussions. The results (16) and (18) indicate that the velocity profile will flatten from that of the viscous flow due to the presence of the constant λ ; and for small values of λ , the result agrees with that of viscous flow under similar assumption made in this case. This is expected as the elasticity of the fluid damps the flow. Again from the result (19) we find that $\alpha\beta c(\cosh \xi_0 - \cosh \xi)$ is large in the core of the liquid, so that the second term is very small, but near the boundary it is appreciable. The motion has therefore the boundary layer character.

Received March 13, 1981

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ASUPRA CURGERII NEPERMANENTE A UNUI FLUID VÎSCOELASTIC DE TIP MAXWELL PRINTR-UN TUB CU SECȚIUNE ELIPTICĂ

(Rezumat)

Se studiază mișcarea unui fluid viscoelastic de tip Maxwell printr-un tub cu secțiunea eliptică sub acțiunea unui gradient de presiune periodic. Soluția problemei se obține prin funcții Mathieu; se cercetează cazurile când frecvența are valoare mică și respectiv valoare ridicată.

RADIAL FLOW OF AN INCOMPRESSIBLE MICROPOLAR FLUID BETWEEN TWO STATIONARY, UNIFORMLY POROUS DISCS

BY

VALERIU AL. SAVA

1. Introduction. Particular flows with suction and injection of micropolar fluids [1] have been investigated in recent years. Plane parallel flow between two infinite plates with constant suction and injection has been considered by Singh and Smith [2] and flows with variable injection along a moving rigid wall by Guram and Smith [3].

In this paper we consider the problem of flow between two co-axial rotating permeable discs. The equations of motion are non-linear and their solution will vary greatly with the magnitudes of the angular velocities of the discs and with the magnitudes of the velocities of suction or injection at the discs. It is likely that many papers will appear discussing this problem and it is logical to divide the research as follows:

- i) Rotation is dominant and injection is negligible. This has been studied by Datta and Sastry [4];
- ii) Injection is dominant and rotation is negligible which will be discussed in the present paper;
- iii) The flow when the effects of rotation and injection are of the same order which will be reported in a later paper.

2. Equations of Motion. The equations of motion of an isotropic micropolar fluid are as follows [1]:

$$(1) \quad \begin{aligned} (\lambda + \mu) \nabla \nabla \cdot v + (\mu + \kappa) \nabla^2 v + \kappa \nabla \times v - \nabla p - \rho f &= \rho \dot{v}, \\ (\alpha + \beta) \nabla \nabla \cdot v + \gamma \nabla^2 v + \kappa \nabla \times v - 2\kappa v + \rho l &= \rho \dot{j}, \end{aligned}$$

where v denotes velocity, v the micro-rotation or spin, p the thermodynamic pressure, f and l the body-force and -couple per unit mass, ρ the density and j the micro-inertia; λ , μ , κ , α , β and γ are material constants; the dot signifies material differentiation with respect to time. Thermal effects have been neglected.

It will be assumed that the motion is symmetric about a central axis Oz which is perpendicular to both discs. Choose cylindrical polar coordinates r , θ and z , where r denotes the radial distance from the axis, θ the polar angle and z the axial distance between the discs.

The planes $z=-d$ and $z=d$ are identified with the planes of the infinite discs, so that $2d$ is the channel width. Through these porous discs fluid is injected or extracted normally with constant velocity V at $z=d$ and $-V$ at $z=-d$.

Setting $v=(u(r, z), 0, w(r, z))$, $v=(0, rf(z), 0)$ the equations (1) for steady axisymmetric flow of an incompressible micropolar fluid reduce to

$$(2) \quad \begin{aligned} (\mu + \kappa) \left(\nabla^2 u - \frac{u}{r^2} \right) - \kappa r f'(z) - \frac{\partial p}{\partial r} &= \rho \left(u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} \right); \\ (\mu + \kappa) \nabla^2 w + 2\kappa f(z) - \frac{\partial p}{\partial z} &= \rho \left(u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right), \\ \gamma r f''(z) + \kappa \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right) - 2\kappa r f(z) &= \rho j(uf(z) + wrf'(z)), \end{aligned}$$

$$(3) \quad \frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{\partial w}{\partial z} = 0,$$

where the prime denotes differentiation with respect to z and

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

The boundary conditions at each disc are

$$(4) \quad \begin{aligned} u=0, \quad w=V, \quad f(z)=0 \quad \text{on} \quad z=d, \\ u=0, \quad w=-V, \quad f(z)=0 \quad \text{on} \quad z=-d. \end{aligned}$$

It has been assumed that the normal fluid velocities at each disc are constant and consequently, similar solutions are sought having the property that flow in the axial direction is uniform over planes parallel to the discs, that is, $w=h(z)$.

Thus from continuity equation (3) it follows that

$$u = -\frac{r}{2} h(z),$$

where the condition that u is finite at $r=0$ has been imposed. Substituting w and u in (2) yields that the pressure must satisfy

$$(5) \quad \frac{\partial p}{\partial z \partial r} = 0,$$

while $h(z)$ and $f(z)$ satisfy

$$(6) \quad \begin{aligned} (\mu + \kappa) h^{IV} - \rho h h''' + 2\kappa f'' &= 0, \\ \gamma f'' - \kappa \left(\frac{h''}{2} + 2f \right) &= \rho j(f'h - h'f). \end{aligned}$$

The non-dimensional variables η , $\bar{h}$, $\bar{f}$, $\bar{j}$ are now introduced as

$$(7) \quad \eta = \frac{z}{d}, \quad \bar{h} = \frac{h}{V}, \quad \bar{f} = \frac{d^2 f}{V}, \quad \bar{j} = \frac{j}{d^2}$$

so that the Eqs. (6) become

$$(8) \quad \left(\frac{1}{R_\mu} + \frac{1}{R_x} \right) \bar{h}^{iv} - \bar{h} \bar{h}^{''''} + \frac{2}{R_x} \bar{f}'' = 0,$$

$$\frac{1}{R_\gamma} \bar{f}'' - \frac{1}{2R_x} \bar{h}'' - \frac{2}{R_x} \bar{f} = \bar{j}(\bar{f}\bar{h} - \bar{f}\bar{h}'),$$

where the prime denotes now differentiation with respect to η , and

$$(9) \quad R_\mu = \frac{\rho d V}{\mu}, \quad R_x = \frac{\rho d V}{\kappa}, \quad R_\gamma = \frac{\rho d^3 V}{\gamma}$$

are the suction Reynolds numbers.

The Eqs. (8) have to be solved subject to the boundary conditions

$$(10) \quad \begin{aligned} \bar{h}' &= 0, \quad \bar{h} = 1, \quad \bar{f} = 0 & \text{on } \eta = 1, \\ \bar{h}' &= 0, \quad \bar{h} = -1, \quad \bar{f} = 0 & \text{on } \eta = -1. \end{aligned}$$

When the velocity profile has been obtained the pressure distribution can be calculated from

$$(11) \quad p(r, \eta) = p(0, -1) + \frac{V^2}{2} \rho \left\{ 1 - \bar{h}^2(\eta) + \left(\frac{1}{R_\mu} + \frac{1}{R_x} \right) \left(2\bar{h}'(\eta) - \frac{c r^2}{2d^2} \right) \right\} -$$

$$- \kappa V d \int_{-1}^{\eta} \bar{f}(\eta) d\eta,$$

where c is a constant and $p(0, -1)$ is a reference pressure evaluated at the centre of the lower disc.

3. Solution for Small Suction Reynolds Numbers. Suppose R_μ , R_x , R_γ are small (it is the case of small suction or injection rate), and of the same order.

We set

$$(12) \quad R_\mu = c_x R_x = c_\gamma R_\gamma = \varepsilon,$$

where $c_x = \kappa/\mu$, $c_\gamma = \gamma/d^2\mu$.

The solutions of (8) and (10) are taken in the form

$$(13) \quad \begin{aligned} \bar{h}(\eta) &= h_0(\eta) + \sum_{i=1}^{\infty} h_i(\eta) \varepsilon^i, \\ \bar{f}(\eta) &= f_0(\eta) + \sum_{i=1}^{\infty} f_i(\eta) \varepsilon^i. \end{aligned}$$

Substituting (13) into (8), and equating the coefficients of the same powers of ε , we obtain the systems

$$(14) \quad \begin{cases} (1+c_x)h_0^{IV}+2c_x f_0''=0, \\ 2c_x f_0''-4c_x f_0-c_x h_0''=0, \end{cases}$$

$$(15) \quad \begin{cases} (1+c_x)h_i^{IV}+2c_x f_i''=\sum_{n=0}^{i-1} h_n h_{i-1-n}'', \\ 2c_x f_i''-4c_x f_i-c_x h_i''=\sum_{n=0}^{i-1} (h_n f_{i-1-n}'-h_n' f_{i-1-n}), \quad i \geq 1. \end{cases}$$

The boundary conditions (10) now assume the form

$$(16) \quad \begin{aligned} h_i'(-1) &= h_i'(1) = 0, & i \geq 0; \\ h_0(-1) &= -1, & h_0(1) = 1; \\ h_i(-1) &= h_i(1) = 0, & i \geq 1; \\ f_i(-1) &= f_i(1) = 0, & i \geq 0. \end{aligned}$$

The appropriate solution of (14) satisfying the boundary condition (16) is

$$(17) \quad \begin{aligned} h_0 &= \eta - A_0 \left[\eta^3 - \left(-1 \frac{c_x}{2+c_x} \right) \eta - \frac{c_x}{c_x+2} \frac{\sinh(a\eta)}{\sinh a} \right], \\ f_0 &= \frac{3}{4} \left(\frac{\sinh(a\eta)}{\sinh a} - \eta \right), \quad \text{where} \quad A_0 = \left[2 + \frac{c_x}{c_x+2} - \frac{ac_x}{2+c_x} \frac{\cosh a}{\sinh a} \right]^{-1}. \end{aligned}$$

The solution of (15) when $i=1$ is

$$(18) \quad \begin{aligned} h_1(\eta) &= C\eta + 2 \frac{3A_3+2A_2+A_1}{\sinh a - a \cosh a} \sinh(a\eta) + \eta^2 (A_3\eta^5 + A_2\eta^3 + A_1\eta) - \\ &\quad - \frac{B}{1+c_x} \left[\eta^3 + \frac{2\sinh(a\eta)}{\sinh a - a \cosh a} - \left(1 + \frac{2\sinh a}{\sinh a - a \cosh a} \right) \eta \right], \\ f_1(\eta) &= \frac{3}{4} \left(\eta^3 - \frac{5}{2} \eta^5 \right) - 2(1+c_x)a^2 \frac{3A_3+2A_2+A_1}{\sinh a - a \cosh a} \sinh(a\eta) - \\ &\quad - (1+c_x)(42A_3\eta^5 + 20A_2\eta^3 + 6A_1\eta) + B \left(\frac{3c_x}{1+c_x} \eta + \frac{2a^2 \sinh(a\eta)}{\sinh a - a \cosh a} \right), \end{aligned}$$

where

$$\begin{aligned} A_1 &= \frac{1}{a^2(2+c_x)} \left(\frac{j}{2} + \frac{c_x}{c_x} + 1 \right) - \frac{75}{a^4(2+c_x)} - \frac{3c_x}{4c_x(2+c_x)}, \\ A_2 &= \frac{1}{20(2+c_x)} \left(\frac{j}{2} + \frac{c_x}{c_x} + 1 \right) - \frac{15}{4(2+c_x)a^3}, \end{aligned}$$

$$A_3 = -\frac{5}{56(2+c_x)},$$

$$B = -\frac{9}{8} + 2(1+c_x)a^2 C \sinh a + (1+c_x)(42A_3 + 20A_2 + 6A_1) \times$$

$$\times \left(\frac{3c_x}{1+c_x} + \frac{2a^2 \sinh a}{\sinh a - a \cosh a} \right)^{-1},$$

$$C = -(A_3 + A_2 + A_1) - \frac{6A_3 + 4A_2 + 2A_1}{\sinh a - a \cosh a} \sinh a,$$

$$a^2 = \frac{c_x(c_x + 2)}{c_x(c_x + 1)}.$$

Further terms of the series solution can be obtained but clearly they become increasingly complicated.

In the case of large suction Reynolds numbers it results immediately that the microstructure of fluid has no contribution to flow.

Received March 1, 1982

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CURGAREA RADIALĂ A UNUI FLUID MICROPOLAR INCOMPRESIBIL ÎNTRE DOUĂ DISCURI POROASE STAȚIONARE

(Rezumat)

Se rezolvă problema curgerii unui fluid micropolar incompresibil între două discuri coaxiale când pe un disc are loc o injecție sau o extracție de fluid.

EXACT SOLUTIONS IN THE LINEARIZED THEORY OF SIMPLE
FLUIDS WITH MEMORY

BY

C. FETECĂU and CORINA FETECĂU

1. Introduction. The linearized constitutive equations of a heat conducting incompressible simple fluid, as they were deduced in [1], are

$$\begin{aligned}
 (1.1) \quad T &= -pI + S = -pI - 2\rho \int_0^\infty \zeta'(s) G(s) ds, \\
 \bar{q} &= -k(0)\bar{g} - \int_0^\infty k'(s)\bar{g}^t(s) ds, \quad (\bar{g}^t(\cdot) = F^t(\cdot)^T \bar{g}^t(\cdot)), \\
 \psi &= \psi_0 + \beta(0)\theta + \int_0^\infty \beta'(s)\theta^t(s) ds + \int_0^\infty \zeta'(s) \text{tr} G(s) ds, \\
 \eta &= -\beta(0), \quad (G(\cdot) = F^t(\cdot)^T F_t(\cdot) - I),
 \end{aligned}$$

where T , S , $\bar{q}$, ψ and η are the stress tensor, the extra stress tensor, the heat flux, the free energy and the entropy at time t ; p is the undetermined spherical stress; $F^t(\cdot)$, $\theta^t(\cdot)$ and $\bar{g}^t(\cdot)$ are the histories of the relative deformation gradient, of the temperature and of the temperature gradient; $\zeta(\cdot)$, $k(\cdot)$ and $\beta(\cdot)$ are material relaxation functions and ψ_0 is a material constant.

The aim of this work is to present some exact solutions corresponding to the curvilinear flows of simple fluids whose constitutive equations are given by (1.1).

2. General Curvilinear Flows. Let us consider the flow of a fluid whose velocity with respect to an orthogonal curvilinear co-ordinate system x^i ($i=1, 2, 3$), to be all time tangent to the surfaces $x^1=\text{constant}$. If, further, the tangential components of the velocity of this flow depend only of x^1 , i.e.

$$(2.1) \quad \dot{x}^1=0, \quad \dot{x}^2=\omega(x^1, t), \quad \dot{x}^3=\mathcal{O}(x^1, t),$$

then the above surfaces will be rigid material surfaces.

The flows, so defined, will be called (see [2]) „general curvilinear flows“. For them, the trajectory of a fluid particle will belong to the same

surface $x^1 = \text{constant}$. In the following we shall keep our attention to two such flows.

a) *The Couette's plane flow*. It is known that by the Couette's plane flow one means the motion of a fluid between two parallel plane plates. We choose the Cartesian co-ordinate system $Oxyz$ so that the two plates be in the planes $x=0$ and $x=h$, the first of them being fixed and the other moving in the Oz direction. Clearly, the planes $x = \text{constant}$ will be rigid material surfaces.

For these motions, the continuity equation is automatically satisfied, while the motion equations and energy balance take the forms

$$(2.2) \quad \partial_x S_{xx} - \rho \partial_x \varphi = 0, \quad \partial_x S_{xy} - \rho \partial_y \varphi = 0, \quad \partial_x S_{xz} - \rho \partial_z \varphi = \rho \partial_t \mathcal{O}$$

and

$$(2.3) \quad S_{xx} \partial_x \mathcal{O} - \partial_x q_x + \rho r = \rho \partial_t \varepsilon,$$

where $\varphi = p/\rho + v$ ($-\text{grad } v$ is the body force), r is the heat supply and $\varepsilon = \psi + \theta \eta$ the internal energy.

The solutions of the system (2.2)–(2.3) in the assumptions:

1. The fluid adheres to the plates;
2. Its flow is steady (the functions $\mathcal{O}$ and θ depending of x only), the plate $x=h$ moving with the constant velocity V in the Oz direction;
3. The heat supply is a negligible quantity ($r=0$), and the two plates are maintained at the constant temperature $\theta=0$; are of the form

$$(2.4) \quad \mathcal{O}(x) = -\frac{a}{2f} x^2 + \frac{b}{f} x, \quad S_{xx} = az + c(t) + \rho \varphi;$$

$$(2.5) \quad \theta(x) = -\frac{1}{k(\infty)f} \left(\frac{a^2}{12} x^4 - \frac{ab}{3} x^3 + \frac{b^2}{2} x^2 dx \right),$$

where $b = (ah^2 + 2fV)/2h$, $f = 2\rho \int_0^\infty s \zeta'(s) ds$, $c(t)$ is an arbitrary function of t , $d = h \left(\frac{a^2 h^2}{12} - \frac{ab h}{3} + \frac{b^2}{2} \right)$ and a is the driving force per unit volume in the z direction [3]. This driving force is a measurable quantity.

The relations (2.4) and (2.5) together with (1.1)_{1,2} enable us to determine the components of the stress tensor and of the heat flux.

Such, we have

$$(2.6) \quad T_{xx} = az + c(t) + \rho v, \quad T_{xy} = T_{yz} = 0,$$

$$T_{xz} = -ax + b, \quad T_{yy} = T_{zz} = -p$$

and

$$(2.7) \quad q_x = \frac{1}{f} \left(\frac{a^3}{3} x^3 - abx^2 + b^2 x - d \right), \quad q_y = q_z = 0,$$

where $g = -2\rho \int_0^\infty s^2 \zeta'(s) ds$ is also a material constant.

Furthermore, (1.1)₁, (2.6) and (2.4) lead to

$$(2.8) \quad p = -az - c(t) + (-ax + b)^2 \frac{g}{f^2} - \rho v$$

and

$$(2.8') \quad Q = \int_0^h \mathcal{O}(x) dx = \frac{h^2}{2f} \left(h - \frac{ah}{3} \right),$$

where Q is the volume discharge per unit time through a cross-section of the channel one unit deep.

Remark 1. As the volume discharge Q , the normal stresses $T_{xx}(0)$ and $T_{xx}(h)$ necessary to be applied to the plates in order to maintain them at the constant distance h and temperature θ is an interior point of the fluid, are measurable quantities, it results that all material constants may be determined.

Remark 2. The stress state of the fluid is such determined up to a hydrostatic pressure of time $c(t)I$.

Remark 3. If the normal thrust (per unit area) on one of the bounding plates is known as a function of t for some value of y and z , then T_{xx} in (2.6)₁ can be set equal to that thrust and $c(t)$ can be calculated. Then, T_{xx} , T_{yy} and T_{zz} (see (2.6) and (2.8)) may be calculated at any point in space.

b) *Axi-symmetrical flows.* These motion can occur between two infinite coaxial cylinders, which rotate about their common axis with the constant velocities Ω_1 and Ω_2 and slide in the direction of the same axis with the velocities V_1 and V_2 . In a cylindrical co-ordinate system (r, φ, z) in which $r=0$ is the axis of the cylinders, the surfaces $r=\text{constant}$ are rigide material surfaces.

Denoting by R_1 and R_2 ($R_1 < R_2$) the radii of the two cylinders and assuming again the conditions 1–3 to be satisfied, we can reduce our problem to the system

$$(2.9) \quad \begin{aligned} \left(\partial_r^2 + \frac{3}{r} \partial_r \right) \omega(r) &= 0, & \omega(R_1) &= \Omega_1, & \omega(R_2) &= \Omega_2; \\ \left(\partial_r^2 + \frac{1}{r} \partial_r \right) \mathcal{O}(r) + \frac{a}{f} &= 0, & \mathcal{O}(R_1) &= V_1, & \mathcal{O}(R_2) &= V_2; \\ \left(\partial_r^2 + \frac{1}{r} \partial_r \right) \theta(r) + \frac{f}{k(\infty)} [r^2 (\partial_r \omega)^2 + (\partial_r \mathcal{O})^2] &= 0, \\ \theta(R_1) &= \theta(R_2) = 0. \end{aligned}$$

Taking into account the fact that

$$\left(\partial_r^2 + \frac{3}{r} \partial_r \right) \cdot = \frac{1}{r} \partial_r \left[\frac{1}{r} \partial_r (r^2 \cdot) \right] \quad \text{and} \quad \left(\partial_r^2 + \frac{1}{r} \partial_r \right) \cdot = \frac{1}{r} \partial_r (r \partial_r \cdot)$$

we obtain the following solutions

$$(2.10) \quad \omega = \frac{R_1^2 R_2^2 (\Omega_2 - \Omega_1)}{r^2 (R_1^2 - R_2^2)} + \frac{\Omega_1 R_1^2 - \Omega_2 R_2^2}{R_1^2 - R_2^2},$$

$$(2.11) \quad \mathcal{O} = -\frac{a}{4f} (r^2 - R_2^2) + b \ln \frac{r}{R_2} + V_2$$

and

$$(2.12) \quad \theta = -\frac{f}{k(\infty)} \left[\frac{a^2}{64f^2} (r^4 - R_2^4) - \frac{ab}{4f} (r^2 - R_2^2) + \frac{b^2}{2} \ln(rR_2) \ln\left(\frac{r}{R_2}\right) + \right. \\ \left. + \frac{R_1^4 R_2^2 (\Omega_1 - \Omega_2)^2}{(R_1^2 - R_2^2)^2} \frac{R_2^2 - r^2}{r^2} \right] + d \ln\left(\frac{r}{R_2}\right),$$

where a has the same significance as above,

$$b = \frac{4f(V_2 - V_1) + a(R_2^2 - R_1^2)}{4f \ln(R_2/R_1)}$$

and

$$d = \frac{f}{k(\infty) \ln(R_1/R_2)} \left[\frac{a^2}{64f^2} (R_1^4 - R_2^4) - \frac{ab}{4f} (R_1^2 - R_2^2) + \right. \\ \left. + \frac{b^2}{2} \ln(R_1 R_2) + \ln\left(\frac{R_1}{R_2}\right) + \frac{R_1^4 R_2^2 (\Omega_1 - \Omega_2)^2}{R_2^2 - R_1^2} \right].$$

The components of the stress tensor and of the heat flux are

$$(2.13) \quad T_{rr} = -p + g \left[\frac{4R_1^4 R_2^4 (\Omega_2 - \Omega_1)^2}{r^4 (R_1^2 - R_2^2)^2} + \left(\frac{a}{2f} r + \frac{b}{r} \right)^2 \right], \\ T_{r\varphi} = \frac{2f R_1^2 R_2^2 (\Omega_2 - \Omega_1)}{r^2 (R_1^2 - R_2^2)}, \quad T_{rz} = -\frac{a}{2} r + \frac{b}{r} f,$$

$$(2.14) \quad T_{\varphi z} = 0, \quad T_{\varphi\varphi} = T_{zz} = -p, \\ q_r = f \left[\frac{a^2}{16f^2} r^3 - \frac{ab}{2f} r + \frac{b^2 \ln r}{r} - \frac{2R_1^4 R_2^4 (\Omega_1 - \Omega_2)^2}{r^3 (R_1^2 - R_2^2)^2} \right] - k(\infty) \frac{d}{r}, \quad q_\varphi = q_z = 0$$

and

$$p = -az - c(t) + g \left[\frac{4R_1^4 R_2^4 (\Omega_2 - \Omega_1)^2}{r^4 (R_1^2 - R_2^2)^2} + \left(\frac{a}{2f} r + \frac{b}{r} \right)^2 \right] + \\ + g \left[-\frac{R_1^4 R_2^4 (\Omega_2 - \Omega_1)^2}{r^4 (R_1^2 - R_2^2)^2} + \frac{a^2}{8f^2} r^2 + \frac{ab}{f} \ln r - \frac{b^2}{2r^2} \right] + \\ + \rho \left[-\frac{R_1^4 R_2^4 (\Omega_2 - \Omega_1)^2}{2r^2 (R_1^2 - R_2^2)^2} + \frac{2R_1^2 R_2^2 (\Omega_2 - \Omega_1) (\Omega_1 R_1^2 - \Omega_2 R_2^2)}{(R_1^2 - R_2^2)^2} \ln r + \right. \\ \left. + \left(\frac{\Omega_1 R_1^2 - \Omega_2 R_2^2}{R_1^2 - R_2^2} \right)^2 \frac{r^2}{2} \right] + \rho v.$$

As all material constants are also measurable we can consider our problem completely solved.

Remark 4. If the two cylinders rotate with the same velocity $\Omega_1 = \Omega_2 = \Omega$, then all particles of the fluid will rotate with the same velocity Ω .

Remark 5. If the driving force $a=0$ and $V_1=V_2=V$, then all particles of the fluid are moving in the direction of the axis with the same velocity V .

Remark 6. If further $a=0$, $\Omega_1=\Omega_2=\Omega$ and $V_1=V_2=V$, then all particles of the fluid are moving in the same way at the same constant temperature $\theta=0$. Furthermore, the tension in each point reduces to a hydrostatic pressure and the heat flux is zero.

Remark 7. The solutions corresponding to the Couette's flow through a right circular cylinder can be obtained from the above by making $R_1 \rightarrow 0$ in the expressions of b , d and then ω , $\mathcal{O}$, θ and so on. For ω , $\mathcal{O}$ and θ we have for example

$$(2.15) \quad \omega(r)=\Omega, \quad \mathcal{O}(r)=\frac{a}{4f}(R^2-r^2)+V$$

and

$$(2.16) \quad \theta(r)=\frac{a^2}{64f^2k(\infty)}(R^4-r^4),$$

where R is the radius of the cylinder and Ω and V its velocities of rotation and sliding.

Received December 10, 1983

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SOLUȚII EXACTE ÎN TEORIA LINIARIZATĂ A FLUIDELOR SIMPLE CU MEMORIE

(Rezumat)

Se dau câteva soluții exacte corespunzând curgerii curbilinii a unui fluid simplu, având ecuațiile constitutive date de (1.1).

STUDIES ON THE REALIZATION OF CdS AND Cu_2S THIN PHOTOCONDUCTOR FILMS BY LIQUID PHASE REACTION METHOD

BY

I. BURSUC, GH. ZET, MAGDA GHERGHEL, I. ROȘCA,
EMILIANA GHERASIMESCU and IOANA MIHAI

1. Introduction. It is well known that the research and development program in the energetic field includes, besides the development of classical energy sources, the utilization of non-conventional new energy sources, like those based on the photovoltaic effect. In this connection, a special attention is given to the photocells made of photoconductor thin layers with heterojunction. A very important problem concerning these photocells is to improve their efficiency (by improving their parameters and working conditions) and to lower their cost. But the increase of the efficiency implies, in most cases, the utilization of special technologies and materials, that influences the cost of the photocells and restricts their utilization possibility. Thus, the realization of CdS/ Cu_2S thin films photocells with heterojunction, by means of vacuum evaporation needs a long discontinuous costly technological process and a high consumption of energy to produce a high vacuum [1]...[3], [10]. At the same time, when using the vacuum deposition it is very difficult to control and ensure the stoichiometry of cadmium sulphide CdS which depends on the partial pressure of Cd and S vapours, not so easily to be measured during the deposition [1], [2], [4].

In order to simplify the production process of these photocells by reducing the duration of the technological process and lowering their cost, we shall study the possibility of getting them by simple methods and using cheaper and readily available materials. We want to obtain thus a CdS/ Cu_2S heterojunction cell by producing thin films by means of liquid phase reaction method [1]... [3], [10].

2. Production Technology of Thin Photoconductor Films by Liquid Phase Reaction. *Experimental Results.* The liquid phase reaction method usually consists [1]...[3], [10] in sputtering, on previously heated substrates (at temperatures convenient for the corresponding chemical reactions), solutions of heavy metals salts, $A_{II}B_{IV}$, small amounts of organic agents (thiourea, N, N dimethylthiourea, selenourea or N, M dimethylselenourea) being introduced in solution; these agents are able to interact with the solutions of the corresponding salts. The used organic agents induce precipitates of the $A_{II}B_{IV}$ compounds, like CdS, Cu_2S when the liquid mixture of the corresponding salts is heated up to a temperature adequate for the chemical reactions. As the result of sputtering the mixture on the previously heated substrate, a chemical reaction takes place which determines the formation of the semiconductor compound as a thin layer of a certain thickness.

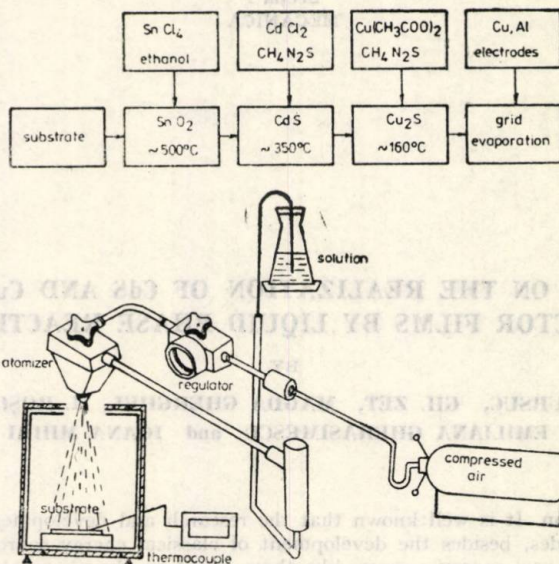


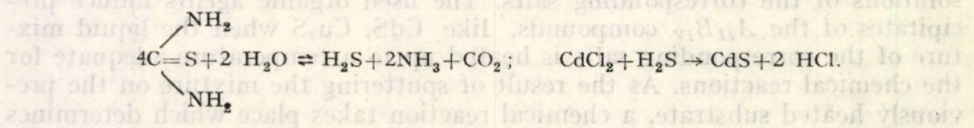
Fig. 1

In Fig. 1, we schematically represent both the technological sequences covered during the experiments and the sketch of the used sputtering device.

As substrates, we have used window glass plates of $8.5 \times 11.5 \times 0.20$ cm, optical glass plates of $8 \times 10 \times 0.15$ cm and usual glass plates of $5 \times 2 \times 0.2$ cm.

Each experimental step aimed at establishing the optimum deposition conditions that should lead, after the sputtering, to the formation of very uniform layers on the used substrates. Thus, we meant to find the optimum values of the sputtering pressure, the distance between the substrate and the pulverizer, the substrate temperature, the solution concentration. We have also followed the way the substrate temperature and solution concentration influence the quality of the obtained semiconductor layer: its thickness, resistivity, transparency.

In order to obtain the CdS layers, we slowly cooled down to about 300°C the samples on which SnO_x thin films were previously formed. The SnO_x films were obtained by the same method of liquid phase reaction [10] and represent the negative electrode of $\text{CdS}-\text{Cu}_x\text{S}$ heterojunction [1], [2], [4]. A solution of CdCl_2 and tiourea ($\text{CH}_4\text{N}_2\text{S}$) in distilled water was then sputtered at a constant temperature, the second substance playing the role of sulphur inductor in the reaction



It was noticed, as the result of numerous experiments, that the CdS layer is generated for a concentration of 4.5...5% CdCl_2 and 5% tiourea,

the substrate being heated at a temperature ranging between 275° and 425°C. Some experimental results are presented in Table 1; they were obtained when producing thin CdS layers by sputtering the solution of CdCl₂ and CH₄N₂S on SnO_x glass substrate, previously heated at corresponding temperature.

Table 1

No	CdCl ₂ concentration, %	P Pa	d cm	t °C	$\bar{\sigma}_{CdS}$ μm	ρ_{CdS} Ω cm. 10 ³
1	6.5	(0.5...1.5)10 ⁵	10...20	425	3.2	0.98
2				400	2.6	1.25
3				375	1.8	1.60
4				350	1.4	1.85
5				325	0.9	1.98
6				300	0.6	2.01
7				275	0.2	2.22
1	5.5	(0.5...1.5)10 ⁵	10...20	425	2.8	1.42
2				400	2.0	1.81
3				375	1.7	2.54
4				350	1.2	2.86
5				325	1.0	3.20
6				300	0.9	3.86
7				275	0.5	4.20
1	4.5	(0.5...1.5)10 ⁵	10...20	425	2.9	1.48
2				400	2.1	1.78
3				375	1.7	1.98
4				350	1.4	2.40
5				352	1.2	2.78
6				300	0.6	3.60
7				175	0.2	3.80

The experiments aiming at the realization of transparent and uniform CdS layers on SnO_x glass substrate have lead us to the following conclusions :

1. The CdS layer is formed on SnO_x glass substrate for a certain concentration of the solution, ranging between 4.5 and 6.5% CdCl₂. Under the experimental conditions, it was noticed that the utilization of solutions with too high or too low concentrations, as compared to the limits of this interval, does not favour the formation of uniform and transparent CdS layers on SnO_x glass substrate. We explained this by the fact that the utilization of a solution of CdCl₂ and CH₄N₂S of an inadequate concentration results in a marked shift of the reaction equilibrium to the right, thus determining the formation of a great amount of H₂S, which favours the formation of uneven opaque CdS spots on the substrate. Hence the concentration of the solution of CdCl₂ and CH₄N₂S has the decisive role in the generation

of CdS layer, the thiourea being the control agent that directs and doses the formation of CdS layer by means of H_2S .

2. We also noticed the very important role played by the temperature in the chemical reaction and the formation of CdS layer. Thus, at temperatures exceeding $400^\circ C$, a strong rightward shift of the equilibrium of the chemical reaction was seen, resulting in the formation of a thick (more than $4\text{ }\mu m$) non-uniform layer of CdS, of a smaller resistivity ($0.52 \cdot 10^3\text{ }\Omega cm$).

On the other hand, a much lower temperature (under $275^\circ C$) does not favour the development of the chemical reaction, i.e. the formation of CdS on the substrate. Hence the existence of an optimum temperature was established, of $350...375^\circ C$, determining the formation on the substrate of a transparent uniform CdS layer, with an adequate thickness $1.4...2\text{ }\mu m$ and resistivity $\rho = (1.5...2)10^3\text{ }\Omega cm$.

It was also found that in order to ensure the quality of CdS layer (its uniformity, transparency, resistivity) a very constant temperature was necessary during the sputtering and the development of the chemical reactions, since a variation of the substrate temperature with only $5...10^\circ C$, had a negative effect on the formation of CdS layer; the avoidance of the temperature gradient was very important in ensuring a high quality CdS layer.

3. The optimum established value of the sputtering pressure was 1.10^5 Pa , a linear dependence $d=f(p)$ being obtained (Fig. 2); the optimum corresponding distance was 15 cm .

Thus, the uniformity of the obtained CdS layers was better for a low sputtering speed (very slow sputtering), the optimum calculated rate being $40\text{ cm}^3/\text{min}$, and the optimum corresponding time was 4.5 min .

4. A comparison of the resistivity of the sputtered CdS layers with that of the CdS vacuum deposited layers (CdS was deposited in vacuum with a purity of 99.8% on glass substrates of $1.5 \times 1 \times 0.2\text{ cm}$ as layers of different thickness, ranging between 0.1 and $3\text{ }\mu m$), showed a remarkable difference, the resistivity of CdS layers prepared by liquid phase reaction method being much higher than that of the vacuum deposited layers, i.e.

$$\rho_{\text{vacuum}} = 0.1...3\text{ }\Omega cm, \quad \rho_{\text{sputt}} = (0.9...2)10^3\text{ }\Omega cm.$$

We explained this by the fact that an excess of Cd appears in CdS vacuum deposited layers (as the result of CdS decomposition), so that the stoichiometry of these layers can no longer be ensured this depending on the deposition rate, substrate temperature and vacuum performed.

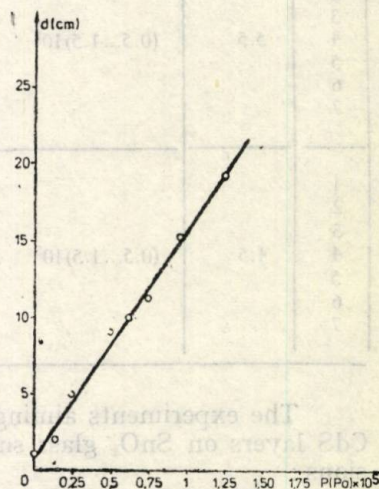
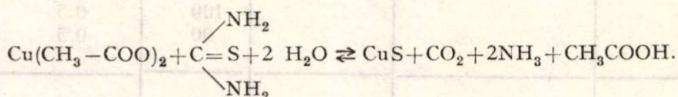


Fig. 2

At the sputtered CdS layers, the stoichiometry is controlled by the chemical reaction itself (by means of tiourea), the deposition rate of CdS being constant and the stoichiometry being not influenced by the concentration of Cd^{2+} and S^{2-} ions, as in the case of CdS vacuum deposited layers [1], [3], [5]. Therefore, as shown in [1], [5], CdS layers obtained by liquid phase reaction method show a crystalline structure with preferential hexagonal phase, the cubic phase being present in a much smaller amount than in the case of vacuum deposited CdS layers. The cubic phase occurs in the CdS vacuum deposited layers as packing defects among the CdS hexagonal crystallites, thus influencing the quality of the deposited layers; it follows that the presence of the cubic phase is not wanted [1], [5].

In the next stage of the experiments, Cu_xS thin layers were produced by sputtering a solution of 4.5...6.5% $\text{Cu}(\text{CH}_3\text{COO})_2$ (copper acetate) and 5% $\text{CH}_4\text{N}_2\text{S}$ in distilled water on glass substrate SnO_x -CdS previously cooled very slowly down to 130°C . The stirring of the mixture of $\text{Cu}(\text{CH}_3\text{COO})_2$ and $\text{CH}_4\text{N}_2\text{S}$ was necessary for a long time (6...7 minutes) at room temperature, in order to ensure the development of the chemical reaction on the substrate



After the formation of CuS layer on the substrate, secondary reactions take place under the influence of the temperature [3], [6], resulting in the production of a small amount of Cu_xS , joining the quantity of CuS; the ratio between the two resulted compounds depends on many factors, like the deposition rate, substrate temperature, concentration of the used solutions, thickness of the preceding SnO_x -CdS layers. We consider that during the formation of Cu_xS a very important role is played by the diffusion, materialized in the ionic exchange that takes place between CdS and $\text{Cu}(\text{CH}_3-\text{COO})_2$ untill the equilibrium is established: $2\text{Cu}^{2+} \rightleftharpoons \text{Cu}^{2+} + \text{Cu}^0$. A high concentration of Cu^{2+} ions will enable the rapid formation of cuprous sulphide in a too high amount, resulting in a thick and non-uniform CuS layer. On the other hand, a slower oxidation of Cu^+ ions leads to the uniform formation of a small amount of CuS together with a thin uniform Cu_xS layer.

The experimental results obtained in this stage (some of which are presented in Table 2) led us to the following conclusions:

1. For the formation of very uniform Cu_xS thin layers, rather important are both the concentration of the employed solutions and the substrate temperature, the two parameters allowing for the shift of (Cu^{2+} - Cu^+) equilibrium toward the generation of a Cu_xS layer of a better quality. Thus, the ratio $|\text{Cu}^{2+}|/|\text{Cu}^+|$ (which, as it is well known [9], is about 10^5 ... 10^6 for the equilibrium in aqueous medium), can be displaced in the wanted direction, depending the temperature and concentration. Thus, one can see that for a smaller concentration of the solution (about 4.5%) and for temperatures ranging between 150 ... 120°C the Cu_xS layer quality (its thickness, resistivity and uniformity) is better, since under these conditions the Cu^{2+} ions prevail, the oxidation of Cu^+ ions occurring slower.

Table 2

No.	$\text{Cu}(\text{CH}_3\text{COO})_2$ concentration %	P Pa	d cm	t °C	Cu_xS μm	$\rho_{\text{Cu}_x\text{S}}$ Ωcm
1	6.5	$(1...1.5) \cdot 10^5$	15...25	200	0.5	$4.3 \cdot 10^{-2}$
2				175	0.4	$5.6 \cdot 10^{-2}$
3				150	0.2	$9.1 \cdot 10^{-2}$
4				140	0.2	$1.0 \cdot 10^{-1}$
5				120	0.1	$3.0 \cdot 10^{-1}$
6				100	0.3	$4.0 \cdot 10^{-1}$
7				90	0.4	$6.2 \cdot 10^{-1}$
1	5.5	$(1...1.5) \cdot 10^5$	15...25	200	0.5	$3.3 \cdot 10^{-2}$
2				175	0.5	$2.9 \cdot 10^{-2}$
3				150	0.3	$6.8 \cdot 10^{-2}$
4				140	0.3	$8.8 \cdot 10^{-2}$
5				120	0.2	$1.4 \cdot 10^{-1}$
6				100	0.5	$6.9 \cdot 10^{-1}$
7				90	0.5	$8.4 \cdot 10^{-1}$
1	4.5	$(1...1.5) \cdot 10^5$	15...25	200	0.5	$8.2 \cdot 10^{-2}$
2				175	0.3	$1.1 \cdot 10^{-1}$
3				150	0.2	$1.5 \cdot 10^{-1}$
4				140	0.1	$1.3 \cdot 10^{-1}$
5				120	0.2	$6.3 \cdot 10^{-2}$
6				100	0.3	$2.1 \cdot 10^{-1}$
7				90	0.4	$8.7 \cdot 10^{-1}$

2. From the analyses of all the experimental results, the existence of a close dependence became evident, between the concentration of $\text{Cu}(\text{CH}_3-\text{COO})_2$ solution, the substrate temperature and layer resistivity, so that the optimum values that we found for these parameters were as follows: the temperature $150...120^\circ\text{C}$; the concentration of the solution $4...4.5\%$ $\text{Cu}(\text{CH}_3-\text{COO})_2$; the layer resistivity $5 \cdot 10^{-2}...10^{-1} \Omega\text{cm}$. At the same time, in this stage (of Cu_xS layer generation) we found a linear dependence between the distance and pressure, $d=f(p)$, the optimum sputtering distance being 20 cm and the corresponding pressure $1.5 \cdot 10^5$ Pa. The optimum sputtering time was 2 minutes, for a rate of $30 \text{ cm}^3/\text{min}$.

3. By comparing the Cu_xS thin layers obtained by the liquid phase reaction method with those of the layers produced by vacuum deposition and dipping process [3], [6]...[8], a much better adherence to the substrate was found for Cu_xS sputtered layer. We explained this by the fact that the cuprous acetate solution used for the realization of Cu_xS layers presents a stable structure, as compared to the $\text{CuCl}-\text{NH}_4\text{Cl}$ solution used in the dipping process [6], [7] and by the fact that a small amount of HCl must be used in this last method, having a corrosive action. Therefore, the Cu_xS sputtered layers being more adherent to the substrate, than those obtained by the dipping process [6], [7], there is the possibility of their preparation

on larger areas than in the case of other procedures (vacuum deposition and dipping process) where a good adherence to the substrate implies a small area.

The values of the layer resistivity were determined by means of a probe-type device, using an EG 402 multimeter, and the sample thickness was established with a Linnik interference microscope.

3. Final Conclusions. 1. The deposition of SnO_x — CdS — Cu_xS layers by the liquid phase reaction method has allowed the realization of a CdS — Cu_xS heterojunction, whose structure is schematically represented in Fig. 3. As positive electrode, a vacuum deposited copper grid was used covered with a $0.2\text{ }\mu\text{m}$ Al layer (to avoid the oxidation). Under the best experimental conditions that we have established and presented in this work, we have obtained an open circuit voltage V_{oc} of $190\text{--}250\text{ mV}$, and a short-circuit current density I_{sc} , of 8.5 mA/cm^2 .

2. We were thus able to point out that improving the parameters characterizing the CdS — Cu_xS junction is closely correlated with the parameters determining the quality of SnO_x , CdS and Cu_xS layers (the substrate temperature, solution concentration, sputtering pressure and distance, sputtering time and rate), parameters that had a specific influence on the layer resistivity, thickness, uniformity and transparency (Tables 1 and 2). On the other hand, by means of the two parameters: temperature and concentration (that can influence the equilibrium of the chemical reactions of layer formation toward the wanted direction), one can improve the quality of the layer and hence of the CdS — Cu_xS heterojunction.

A linear dependence was established between the sputtering distance and pressure (Fig. 2).

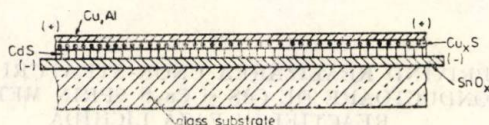


Fig. 3

3. The layer quality was better for substrates of $5 \times 2 \times 0.2\text{ cm}$, greater substrates of $8 \times 10 \times 0.2\text{ cm}$ having a negative influence of the layer uniformity [10]. On the other hand, the optical glass substrates have proved to be less resistant [10], suffering distortions during the heating, resulting in non-uniformities and opacity of the prepared SnO_x and CdS layers.

4. The thin photoconductor layers deposited by liquid phase reaction present the advantage of their realization in a continuous process, with a lower energy consumption and in a shorter time, having a better adherence to the substrate (which allows their production on larger areas) and with a controllable stoichiometry [1], [10]. Their shortcoming consists in their poorer photoconductivity and photosensitivity, as compared to those of the layers obtained by other methods (vacuum deposition, dipping, sintering a.s.o.); yet, their qualities can be improved by controlling some of

the chemical and physical parameters, like temperature, concentration, doping degree. These parameter have a direct influence on the development of the chemical reactions, determining the layer formation on the substrate.

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Received December 20, 1983

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CERCETĂRI PRIVIND REALIZAREA UNOR STRATURI SUBȚIRI FOTOCONDUCTOARE DE CdS și Cu_xS SPRIN METODA REAȚIEI ÎN FAZĂ LICHIDĂ

(Rezumat)

Pe baza datelor experimentale obținute, se scoate în evidență influența condițiilor de depunere (presiune, temperatură, concentrație, tip-suport) asupra procesului de formare a straturilor subțiri de CdS și Cu_xS realizate prin metoda reacției în fază lichidă. Concluziile care au rezultat prezintă importanță în alegerea parametrilor optimi de pulverizare, care să conducă la îmbunătățirea calității straturilor și a heterojuncțiunii CdS— Cu_xS .

STUDIES CONCERNING THE REALIZATION OF SnO_x THIN FILMS FOR PHOTOCONDUCTIVE DEVICES BY MEANS OF THE LIQUID PHASE REACTION METHOD

BY

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1. Introduction. It is well known that the thin photoconductive layers characterized by the photovoltaic effect present nowadays an increasing interest due to their utilization in the production of the photocells — devices accomplishing the conversion of light in electric energy. In this connection, a special attention is given to the photoelements made of thin photoconductive layers with heterojunction structure. The main problem that arises about these photoelements is how to improve their efficiency (by improving their parameters and working characteristics) and to lower their cost. Yet, increasing their efficiency implies, in most cases, the utilization of special materials and technologies, which affects the cost of the photocells, resulting in the limitation of their utilization possibilities. The attempts to lower the cost are materialized in the decrease of the production cost, the automatization and simplification of the heterojunctions production process, the utilization of the cheapest semiconductor materials readily available and which, when conveniently doped, result in an improvement of the photoelements working parameters.

With the view of reducing the cost and duration of photocells production process we proposed to study the possibilities of producing SnO_x thin transparent layers on glass substrate by liquid phase reaction method, layers that should be used as transparent negative electrodes in the $\text{CdS}-\text{Cu}_x\text{S}$ heterojunction photoelements [3], [6], [7]. The method of liquid phase reaction is more simple and rapid and implies a lower energy consumption, as compared to the vacuum deposition method [1], [4], [5], [7]. At the same time, the fact that the SnO_x thin layer made on glass substrate is transparent permits the illumination through the glass (reversed illumination) or even through the both faces of the photoelements (the light can also fall from the side of Cu_xS layer), which leads to the increase of the carrier collection efficiency in Cu_xS layer [3], [6], [7]. The $\text{CdS}-\text{Cu}_x\text{S}$ photoelements whose electrodes are made of SnO_x thin films obtained by liquid phase reaction, are less expensive as compared to the photoelements whose

electrodes are made of vacuum deposited Ag or Au thin layers [1], [3], [4], [7].

2. The Technology of SnO_x Thin Layers Preparation by Liquid Phase Reaction. *Experimental Results.* The method consists in sputtering a solution containing a Sn salt on a substrate previously heated up to a temperature appropriate for the corresponding chemical reaction, the hydrolysis of Sn salt taking place on the substrate and resulting in the formation of the semiconductor compound as a thin layer of a certain thickness. We used as substrates usual window glass plates of $8.5 \times 11.5 \times 0.2$ cm, optical glass plates of $8 \times 10 \times 0.15$ cm and smaller common glass of $5 \times 2 \times 0.2$ cm. The Fig. 1, represents the sketch of the sputtering device used in the experiments.

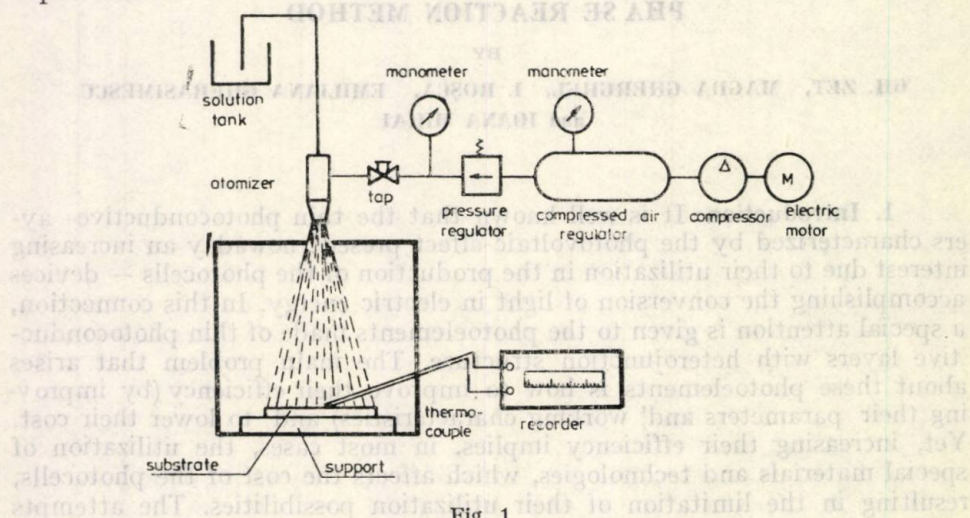
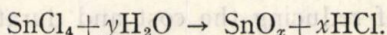


Fig. 1

Thus, for the realization of SnO_x layer, a solution of SnCl_4 and ethanol with 5...6% SnCl_4 was sputtered on substrates previously heated up to 400...600°C. Under the influence of the heating the hydrolysis of SnCl_4 took place, i.e.



We meant to find, in every stage of the experiments, the optimum deposition conditions that should lead, as the result of the sputtering, to the formation on the substrate of as uniform as possible layers. Thus, we proposed to find the optimum values of the sputtering pressure, the distance between the substrate and pulverizer, the substrate temperature, the solution concentration. The stoichiometric composition of SnO_x layer produced by liquid phase reaction being influenced by many factors (the substrate thickness, its temperature and structure, the solution concentration) [1], [2], [5] we investigated the influence of the sputtering pressure and distance, the solution concentration and substrate temperature on the layer parameters: thickness, resistivity, uniformity, transparency. Some experimental results concerning the formation of SnO_x layers are presented in Table 1. These results allowed us to draw the following conclusions:

1) The resistivity of SnO_x layers decreases when increasing their thickness, but they present an opaque aspect.

2) The opacity of these layers was found to diminish when reducing the solution sputtering rate and hence the sputtering pressure, this resulting in a small increase of the resistivity.

3) We noticed that at a too high substrate temperature—exceeding 575°C —the resistivity increases and for temperatures under 575°C it decreases (Table 1). At the same time, it was found that when the temperature decreases under 500°C , even with $25\ldots 30^\circ\text{C}$, the layer resistivity increases. It follows that there is an optimum temperature, favouring the formation on the substrate of the SnO_x layer with adequate thickness and resistivity, making it suitable to be used as electrode (Table 1). Thus, under the experimental conditions, the optimum temperature that we found was of about 475°C , any decrease or increase of the temperature beyond this temperature resulting in a marked increase of SnO_x layer resistivity.

4) The great influence of SnCl_4 solution concentration was established, both on the formation of SnO_x layer and its characteristics: thickness, resistivity, transparency. An optimum concentration of $5\ldots 5.2\%$ SnCl_4 was found.

Table 1

No.	SnCl_4 concentration, %	P Pa	d cm	t $^\circ\text{C}$	g_{SnO_x} μm	$\rho \times 10^{-3} \Omega\text{cm}$
1	6	$(1\ldots 2.75)10^5$	10...30	575	0.48	2.2
2				550	0.50	2.0
3				525	0.52	1.9
4				500	0.58	1.5
5				475	0.65	1.2
6				450	0.78	1.8
7				425	0.92	2.9
1	5.5	$(1\ldots 2.75)10^5$	10...30	575	0.39	1.5
2				550	0.42	1.2
3				525	0.49	1.2
4				500	0.54	1.1
5				475	0.57	0.8
6				450	0.65	1.0
7				425	0.80	2.0
1	5	$(1\ldots 2.75)10^5$	10...30	575	0.41	1.80
2				550	0.45	1.80
3				525	0.50	1.60
4				500	0.53	1.50
5				475	0.55	1.40
6				450	0.62	1.60
7				425	0.75	1.70

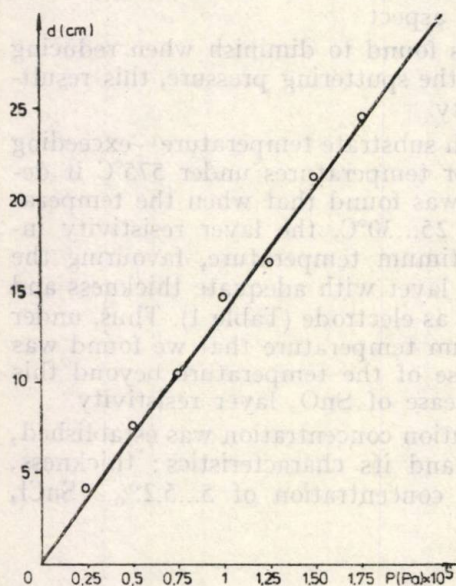


Fig. 1

5) The optimum established sputtering pressure was $1.75 \cdot 10^5$ Pa, the optimum flow rate being about $40 \text{ cm}^3/\text{min}$ and the corresponding optimum sputtering time was 35 sec. The optimum distance between the pulverizer and the sample was 23...24 cm, a linear dependence being established between the distance and the sputtering pressure, $d=f(p)$, that could be a characteristic of the pulverizer used in the sputtering device (Fig. 2).

The layer resistivity was determined with a probe device, using a EG-402 multimeter, and the layer thickness was established with the Linnik interference microscope.

3. Final Conclusions. 1. The experimental results concerning the preparation of SnO_x thin layers by the liquid phase reaction method made evident the special influence of both the substrate temperature and solution concentration, on the layer thick-

ness, resistivity, uniformity and transparency. It was found that a solution concentration exceeding 5.5% SnCl_4 and a temperature higher than 500°C had a negative influence on the layer quality, determining the formation of opaque non-uniform layers with high resistivity. On the other hand, a temperature less than 400°C and a concentration under 4.5% SnCl_4 do not favour the formation of SnO_x layers, resulting only in the spreading of the sputtered solution on the substrate.

2. Therefore, by means of the two parameters, temperature and concentration (that can influence the equilibrium of the chemical reaction in the wanted direction [2], [8]), an improvement can be obtained of the SnO_x layer parameters, that can also influence the quality of $\text{Cu}_x\text{S}-\text{CdS}$ heterojunction [6], [7]. Thus, the optimum values that we have established for the temperature and concentration were $t \simeq 500...475^\circ\text{C}$, $c \simeq 5...5.2\%$ SnCl_4 respectively, the corresponding layers having thicknesses of $0.57...0.60 \mu\text{m}$ and resistivities of $(0.8...0.9)10^{-3} \Omega\text{cm}$.

3. The obtained experimental results made also evident a linear dependence between the sputtering distance and pressure, $d=f(p)$ (Fig. 2).

4. The thin photoconductive layers deposited by the liquid phase reaction method present the advantage of being produced in a continuous way, with a much lower energy consumption and in a shorter time. These layers have a good adherence to the substrate (which allows their realization on larger surfaces as compared to thin layers obtained on small surfaces by the vacuum deposition [3]), and a controllable stoichiometry.

5. The layer quality was better for the substrates of $5 \times 2 \times 0.2$ cm the larger substrates of $0.8 \times 11.5 \times 0.2$ cm having a negative influence on the layer uniformity. At the same time, the substrates made of optical glass have proved to be less resistant (probably due to its chemical composition), suffering distortions during the heating, which resulted in the non-uniformities and marked opacity of the prepared SnO_x layers.

Received December 20, 1983

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CERCETĂRI PRIVIND REALIZAREA STRATURILOR SUBȚIRI DE SnO_x PENTRU DISPOZITIVE FOTOCONDUCTOARE, PRIN METODA REACȚIEI ÎN FAZĂ LICHIDĂ

(Rezumat)

Se prezintă rezultatele cercetărilor experimentale efectuate în scopul obținerii unor straturi subțiri de SnO_x (cu proprietăți de electrod negativ într-o heterojuncțiune $\text{CdS}-\text{Cu}_x\text{S}$) prin metoda reacției în fază lichidă. Sunt prezentate etapele tehnologice, parametrii specifici procesului de pulverizare și condițiile de formare a straturilor subțiri de SnO_x pe suporturi de sticlă de diferite dimensiuni și calități.

RECENZII

BOOK REVIEWS

РЕЦЕНЗИИ

R. BULLOUGH and P. CAUDREY (Eds.), *Topics in Current Physics*. Vol. 17, *Solitons*. Springer Verlag, 1980, 400 pp.

Until 1970 the number of significant exactly soluble problems in physics was limited to a very few: the classical or quantised harmonic oscillator, the linearised many-body problem, the quantised hydrogen atom, Newton's solution of the planetary orbit problem, Onsager's solution of the two-dimensional Ising problem, almost exhaust the list.

The inverse scattering technique discovered in 1967 for solving the Korteweg de Vries equation has now been successfully applied to the solution of a number of nonlinear waves or evolutions of physical significance. Similar techniques are being used in the study of, for example, Yang-Mills field or Einstein's gravitational field equations, and problems in statistical mechanics. The soliton is an important nonlinear elementary excitation — in charge density wave theories of linear conductors, in crystal lattices, and in areas of low temperature physics, for example. It can be a crystal dislocation, a nonlinear plasma excitation, a fluxon, an ultra-short optical pulse or an elementary particle. It has engineered a revolution in all area of nonlinear physics. This is the first book, other than conference reports, devoted to solitons. It is an up-to-date summary of the techniques for finding solutions of the „integrable“ or „near integrable“ nonlinear wave and other systems.

The book will be indispensable to research workers in theoretical and mathematical physics and of interest to workers in many areas of mathematics. It should prove to be the standard text on the theory of solitons and nonlinear waves for some years to come.

Contents: R. Bullough, P. Caudrey, *The Soliton and Its History*. G. L. Lamb Jr., D. W. McLaughlin, *Aspects of Soliton Physics*. R. Bullough, P. L. Caudrey, H. M. Gibbs, *The Double Sine—Gordon Equations: A Physically Applicable System of Equations*. M. Toda, *On a Nonlinear Lattice (The Toda Lattice)*. R. Hirota, *Direct Methods in Soliton Theory*. A. C. Newell, *The Inverse Scattering Transform*. V. E. Zakharov, *The Inverse Scattering Method*. M. Wadati, *Generalized Matrix Form of the Inverse Scattering Method*. F. Calogero, A. Degasperis, *Nonlinear Evolution Equations Solvable by the Inverse Spectral Transform Associated with the Matrix Schrödinger Equation*. S. P. Novikov, *A Method of Solving the Periodic Problem for the KdV Equation and Its Generalizations*. L. D. Faddeev, *A Hamiltonian Interpretation of the Inverse Scattering Method*. A. Luther, *Quantum Solitons in Statistical Physics*.

I. Grosu

W. SCHUMAN and M. DUBAS, *Holographic Interferometry* (Edited by David L. Mac Adam). Springer Verlag, Berlin—Heidelberg—New York, 1979, 194 pp.

The authors treat in a unitary conception the problem of mechanical deformations suffered by opaque bodies using holographic interferometry technique. Being structured into 5 chapters and having a vast bibliography, comprising 366 titles, the book uses a mathematical up-to-date device and accessible for a large category of experts, physicists, research workers, teaching staff interested in undestructive tests.

The first chapter is an introduction which shows that optical interferometry and especially holographic interferometry are some of the most precise methods of measuring with a large field

of application among which the analysis of the mechanical deformations suffered by opaque bodies is included.

In the next chapter, the authors present in an accesible and very clear form the vectorial and tensorial computing elements of differential geometry of the surfaces and kinematics of mechanical deformations used in the explanation and analysis of image formation in holography in the interpretation of defrange panel which is obtained by means of holographic interferometry.

In the 3th chapter there are shown the basic principle of holography as a general method of work, the holographic interferometry with a single exposure, holographic interferometry with a double exposure and holographic interferometry mediated by time, which permit to obtain holographic interferograms, whose interpretation gives the necessary information about various modifications (mechanical deformations) suffered by objects under study.

The way in which the obtained interferograms are interpreted as well as the methods which lead to the extraction of required information from macroscopic frange panels are presented in the 4th chapter. Here are included also all the cases of deformations, starting from the simple to the most complex ones, the authors bringing the contribution to the analysis and interpretation of the fringes when deformation appear, the influence of optical arrangements in the stage of image reconstruction upon fringe localization and their quality. In the work there are also presented new theoretical and practical aspects related to the influence of optical arrangement in getting interferograms, in supplying more complete information, whose analysis is done through differential application of differential operator and through affnors.

Thus, it is demonstrated that holographic interferometry unlike other methods allows not only the determination of vector shifting and tensor breaking in the case of tensionings and mechanical deformations of opaque bodies but also the determinations of the 2nd derivative of shifting, buckling and material coefficient changes.

These parameters complete the information about mechanical deformations of the bodies obtained through the up-to-date technique of holographic interferometry.

Magda Gherghel

Coli tipo 6



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